Sep 30

BFS($s$)

$O(n)$ {
0. $CC[s] = T$ and $CC[u] = F \ \forall u \neq s$
1. $i = 0$
2. $L_0 = \{s\}$
}

3. While $L_i \neq \phi$ ] $T_1$ : be the # of times while loop is run

   $L_{i+1} = \phi \leftarrow O(1)$

   For every $u \in L_i$ ] $T_2$ : be # of itr of for loop (given $i$)

   $T_{12}$: # times algo gets here → for every $(u, w) \in E$ ] $T_3$ : # of iteration of this loop given $i, u$

   $T_{123}$: # time algo gets here → If $CC[w] == F$ }
   $CC[w] = T$ } $O(1)$
   Add $w$ to $L_{i+1}$ }

   $O(1) \rightarrow i{+}{+}$

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Total runtime: $O(n) + T_1 \cdot O(1) + T_{123} \cdot O(1) + T_{12} \cdot O(1)$

$\leq O(n) + T_{123} \cdot O(1) + T_{123} \cdot O(1) + T_{123} \cdot O(1)$

$\leq O(n) + O(T_{123})$

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Goal: $T_{123} \leq O(m)$

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Analysis 1: $T_{123} \leq O(n^3)$ ($\Rightarrow O(n) + O(n^3) = O(n^3)$ overall)

$T_{123} \leq \underset{\leq n}{T_1} * \underset{\leq n}{T_2} * \underset{\leq n}{T_3} \leq n^3$

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Analysis 2: $T_{123} \leq \underset{\leq n}{T_{12}} * \underset{\leq n}{T_3} \leq n^2$ ($\Rightarrow O(n) + O(n^2) = O(n^2)$ overall)

since each $u \in V$ appears in $\leq 1$ layer

Analysis 3: $T_{1\&3} \leq O(m)$ $\quad$ ($\Rightarrow O(n) + O(m) = O(m+n)$ overall)

Input size $N = \Theta(m+n)$

$\Rightarrow$ BFS is a linear time algo.

$$T_{1\&3} = \sum_{i=0}^{T_1} \sum_{u \in L_i} \underbrace{\sum_{(u,w) \in E} 1}_{= n_u}$$

$$= \sum_{i=0}^{T_1} \sum_{u \in L_i} n_u$$

Because each $u \in V$ appears in $\leq 1$ $L_i$ $\rightarrow$

$$\leq \sum_{u \in V} n_u = 2m \quad \blacksquare$$