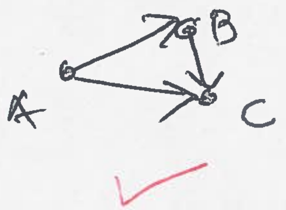
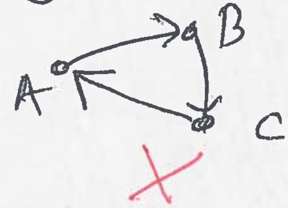
Oct 3

Directed Acyclic Graphs (DAGs)

Def: A directed graph G is a DAG if it has no (directed) cycles





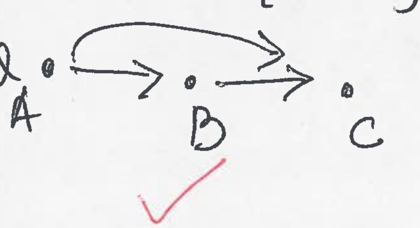
Def: A topological ordering (sorting) of a directed graph is an ordering of vertices

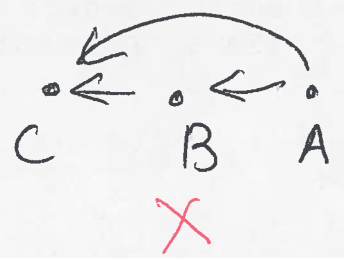
$u_1, \dots, u_n$ s.t.

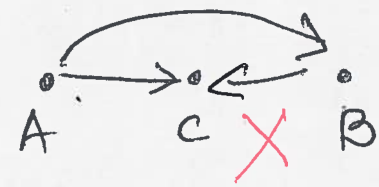
if $(u_i, u_j) \in E \Rightarrow i < j$ $\left[\; u_1 \; u_2 \; \cdots \; u_n \;\right]$

$u_i \longrightarrow u_j$

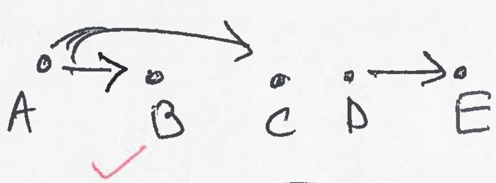
only topological ordering

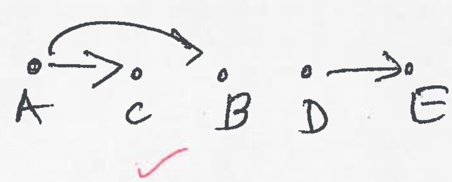


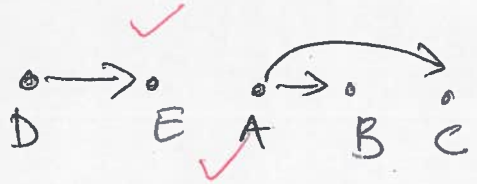


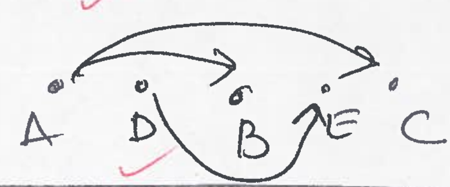


Graph:









Problem:

i/p: A directed graph $G = (V, E)$

o/p: A topological ordering of G (if it $\exists$)

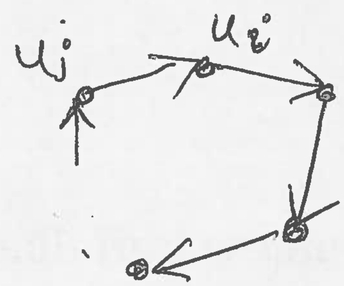
LEMMA 1: Iff G has a topological ordering $\Rightarrow G$ is a DAG.

THEOREM: G is a DAG $\Rightarrow G$ has a topological ordering

Pf (idea) of Lemma 1: By contradiction.

Assume $u_1, u_2, \ldots, u_n$ is a topological ordering of G

BUT G has a directed cycle C



Let i be the smallest index s.t. $u_i \in C$.

$\Rightarrow \exists j$ s.t. $(u_j, u_i) \in E$

C is a cycle. By choice of i, $j > i$.

$\Rightarrow$ contradicts $u_1, \ldots, u_n$ being a topological ordering ☒

Pf of Thm (idea) Via an algo, which given any DAG outputs a topological ordering

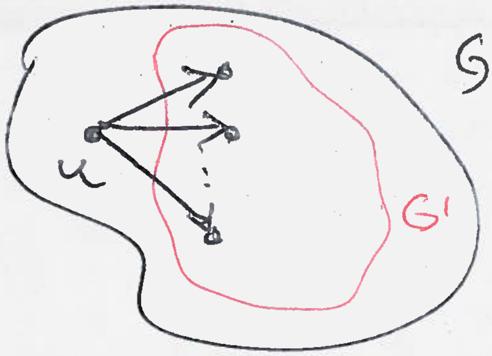
"Reverse engineer" the algo.

Assume: $u_1 \quad u_2 \quad \ldots \quad u_n$ is a topological ordering of G.

Q: How many incoming edges can u_1 have?

A: 0

LEMMA 2: Iff G is a DAG $\Rightarrow \exists\, u \in V$ s.t. u has 0 incoming edges.



G (DAG)

Let $G' = G \setminus \{u\}$

$= (V \setminus \{u\}, E \setminus \{(u,w) \in E \mid w \in V\})$

<u>LEMMA 3:</u> G' is a DAG (Ex)

(since deleting edges ~~cann~~ ~~canno~~ cannot create a new cycle)

Idea: Recurse on G'

TopOrd $(G = (V, E))$

1. If ~~(V)~~ $V = \{u\}$; output u
2. Let w be a vertex with no incoming edges in G
3. $G' = G \setminus \{w\}$
4. Output w; TopOrd (G')

Ex: Pf of correctness (Lemmas 2+3)