$\boxed{\text{Oct 21}}$ **Lemma 2**: Any optimal solution θ' with 0 idle time (but non-zero # inversions), then we can convert it to another optimal solution θ s.t θ has 0 idle time & 0 # inversions.

Notation: idle(S) → idle time of S
#inv(S) → # inversions in S.

Assume: $\#inv(\theta') > 0$.

Pf idea: By "exchange argument"

stop when $\#inv(\theta_{m+1}) = 0$

$$\theta' = \theta_1 \to \theta_2 \cdots \theta_i \to \theta_{i+1} \cdots \to \theta_m \to \theta_{m+1} = \theta$$

$idle(\theta') = 0$
$\#inv(\theta') > 0$

Properties: (i) $idle(\theta_{i+1}) = 0$ · $idle(\theta_i) = 0$
(ii) $\#inv(\theta_i) > 0 \Rightarrow \#inv(\theta_{i+1}) = \#inv(\theta_i) - 1$
(iii) $L(\theta_{i+1}) \leq L(\theta_i)$

✓ $idle(\theta) = 0$
✓ $\#inv(\theta) = 0$
$L(\theta) = L(\theta')$

Idea: Apply the transformation m times ($\#inv(\theta_{m+1}) = 0$)

Note: (i) $idle(\theta_{m+1}) = 0$
(ii) $\#inv(\theta_{m+1}) = 0$
(iii) $L(\theta_{m+1}) \leq \cdots L(\theta_3) \leq L(\theta_2) \leq L(\theta_1) = L(\theta')$
$\Rightarrow L(\theta_{m+1}) \leq L(\theta')$. $\Rightarrow L(\theta_{m+1}) = L(\theta')$
as θ' is optimal.

We want: $\forall i$: $\theta_i \to \theta_{i+1}$ [$\#inv(\theta_i) > 0$ and $idle(\theta_i) = 0$]
$\Rightarrow$ (1) $idle(\theta_{i+1}) = 0$
(2) $\#inv(\theta_{i+1}) = \#inv(\theta_i) - 1$
(3) $L(\theta_{i+1}) \leq L(\theta_i)$

m ∃ s.t $\#inv(\theta_{m+1}) = 0$ since $\#inv(\theta') < n^2$

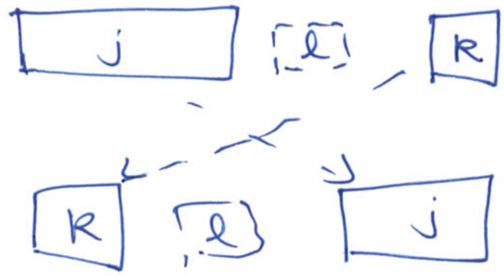
$\# inv(O_i) > 0 \Rightarrow \exists$ an inversion (j,k)

$d_k \quad d_j$

O_i:

Swap j&k!

O_{i+1}:



Obs: $\# inv(O_{i+1}) < \# inv(O_i)$

BUT! lateness of l might also change.

Special case: O_i:

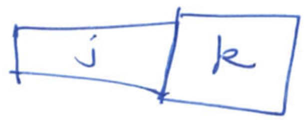
is good

Obs: Special case always $\exists$

Will argue:

Property (a): [as $\# inv(O_i) > 0$] $\exists$ an inv (j,k)

s.t.



$d_k \quad d_j$

Property (b): Swap j & k to get O_{i+1} $\Rightarrow idle(O_{i+1}) = 0$ $\Rightarrow \# inv(O_{i+1}) = \# inv(O_i) - 1$

Property (c): $L(O_{i+1}) \le L(O_i)$

Pf (idea) of (a)

$d_k \quad d_j$

Assume $\exists$ k' between j & k (as o/w we are done)

Case 1: (k',k) is an inversion $\Rightarrow$ done!

Case 2: $d_{k'} \le d_k$ (as (k',k) is not an inversion)

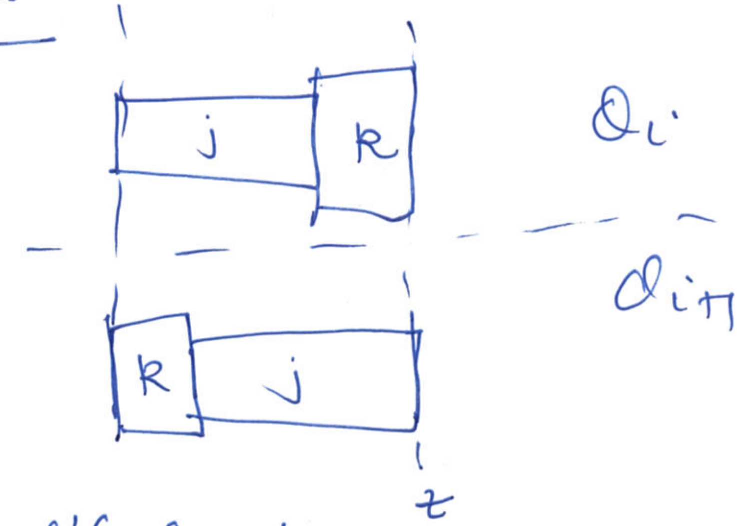
Note: $d_j > d_k$ $\Big\} \Rightarrow d_j > d_{k'} \Rightarrow (j,k')$ is an inversion

$\Rightarrow$ applying b times $\Longrightarrow$ we get

s.t. (j,l) is an inversion.

Pf (idea) of property (c)

~~O_i~~ d_k d_j



$\forall\, m \neq j$ or k $\Rightarrow$ ~~l'(m)~~ $l'_m = l_m$

lateness in O_{i+1} lateness in O_i

Need: $\max(l'_j, l'_k) \leq \max(l_j, l_k)$

Easy case: (A) $l'_k \leq l_k \leq \max(l_j, l_k)$

Other case: It is possible that $l'_j > l_j$

$$l'_j = t - d_j - 1 \qquad d_j > d_k$$

$$< t - d_k - 1 \qquad \Rightarrow -d_j < -d_k$$

$$= l_k$$

(B) $\Rightarrow l'_j < l_k \leq \max(l_j, l_k)$

(A) + (B) $\Rightarrow \max(l'_j, l'_k) \leq \max(l_j, l_k)$.