Oct 26 Dijkstra's algo

$d'(w) = \min_{\substack{u \in R \\ (u,w) \in E}} \{ d(u) + l_{(u,w)} \}$

0. $R = \{s\}$, $d(s) = 0 \leftarrow O(1)$
1. while $\exists\ x \notin R,\ u \in R,\ (u,x) \in E \leftarrow \le n$

   Pick w (among all such vertices x) that $\min d'(w)$ } (**)

   $O(1)$ { Add w to R; $d(w) = d'(w)$ }

Def: Let P_u be the s-u path in Dijkstra trees.

THM: $\forall u \in V$, P_u is a shortest s-u path

($\underset{(Ex)}{\Longrightarrow}$ $d(u)$ is correct)

$\Rightarrow$ Dijkstra's algo is correct.

LEMMA: At the end of each iteration,

$\forall u \in R$, P_u is a shortest s-u path.

LEMMA $\Rightarrow$ THM (apply LEMMA at the end of the algo).

(Ex: Every u s.t $\exists$ s-u path $\Leftrightarrow$ it is in R at the end of the algo

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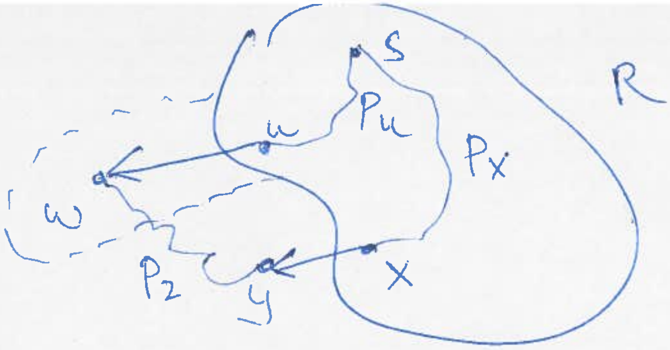
Pf (idea) of LEMMA: By induction on $|R|$

Base case: $R = \{s\}$, $d(s) = 0$ ✓

Inductive hypothesis: Assume true for $|R| = k$ $(k \ge 1)$

Inductive step: Assume algo adds w to R at the end of iteration ($k+1$)



Note: $P_w = P_{u,w}$

Goal: Argue P_w is a shortest path

For contradiction assume $\exists$ an s-w path P'_w s.t. $\ell(P'_w) < \ell(P_w)$ (*)

As $s \in R$ but $w \notin R$

$P'_w = P_x, y, P_2$

$$\ell(P'_w) = \underbrace{\ell(P_x)}_{\text{I.H.} \to \,=} + \ell_{(x,y)} + \underbrace{\ell(P_2)}_{\geq 0} \quad (\text{as } l_e \geq 0\ \forall e \in E)$$

$$\geq d(x) + \ell_{(x,y)}$$

$$\geq d'(y) \geq d'(w) = \ell(P_w)$$

$d'(y) = \min\limits_{\substack{z \in R \\ (z,y)\in E}} \{d(z) + \ell_{(z,y)}\}$

w was added to R

$\Rightarrow \ell(P'_w) \geq \ell(P_w) \Rightarrow$ contradict (*) ∎

Runtime analysis: Keep track of R via a bool array of

→ length n $R[u] = T$ if $u \in R$, $= F$ o/w. Init $R[s] = T$, $R[w] = F\ \forall w \neq$ — $O(n)$

→ Array of length n, $d[u]$ = stores the final answer

$O(n)\{\ d[s] = 0,\ d[w] = \infty\ \forall w \neq s$

Take 1: In every iteration compute $d'(w)$ from scratch.

$$d'(w) = \min_{\substack{u \in R \\ (u,w) \in E}} \{d[u], \ell_{(u,w)}\}$$

for ~~every~~ $w \in V$, look at all possible u s.t. $(u,w) \in E$

$O(|n_w| + 1)\ \{$ compute $d[u] + \ell(u,w)$ & then take min.

Overall $= O\left(\sum_w (|n_w| + 1)\right) = O(m + n)$

⇒ Pick the w that min $d'(w)$ in $O(n)$ time

⇒ each iteration of Dijkstra is $O(m+n)$

⇒ $O(nm+n^2)$ overall.

Take 2: Use a priority Q : n pairs (ID, value)

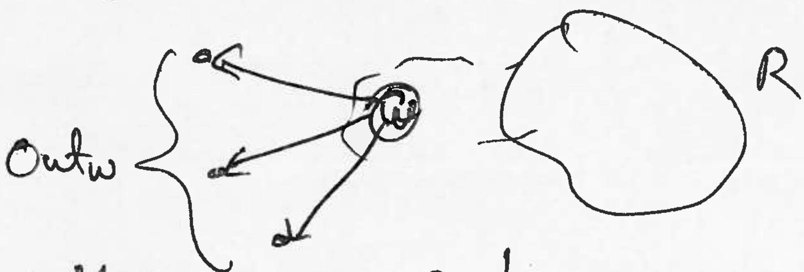
Init : $O(n)$

Extract Min: $O(\log n)$

ChangeValue(ID, v') : $O(\log n)$

In Dijkstra:

$\leq n$ ExtractMin



Only update these Out_w guys ⇒ m ChangeValues

⇒ $O((m+n)\log n)$ overall.