Nov 16

Weighted interval Scheduling

Input: n jobs, i^{th} job (s_i, f_i, v_i)

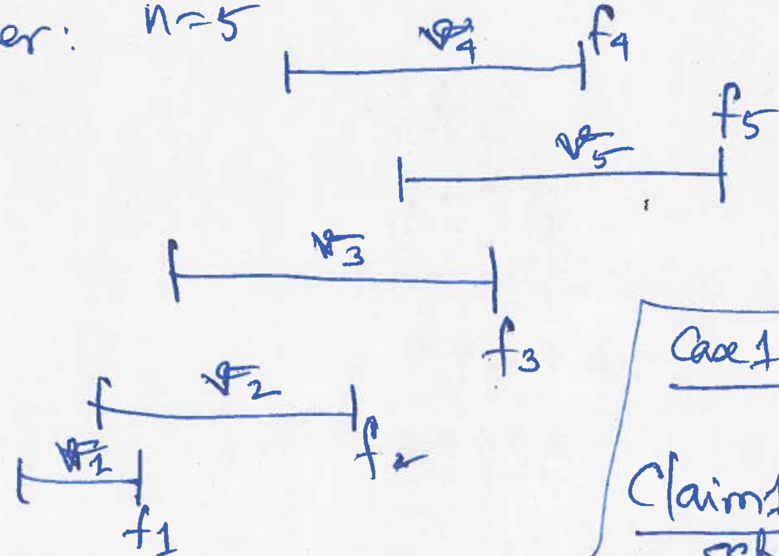
start time → s_i; finish time → f_i; value → v_i

Output: Is a valid schedule $S \subseteq [n]$ that maximizes its value $v(S) = \sum_{i \in S} v_i$

Interval scheduling: all v_i are same (say 1)

Assume: $f_1 \le f_2 \le \dots \le f_n$ [O/w pay $O(n \log n)$ for sorting]

Consider: $n = 5$



Let O be an optimal solution for $[5]$

Case 1: $5 \notin O$

$\Rightarrow O \subseteq [4]$

Claim 1: O is a valid schedule for $[4]$ ← true as O does not have any conflict

Claim 2: O is an optimal schedule for $[4]$.

Pf idea: For contradiction assume $\exists$ valid schedule O' for $[4]$ s.t. $v(O') > v(O)$

$\Rightarrow O'$ is a valid schedule for $[5]$ $\Rightarrow$ contradicts the optimality of O.

Case 2: $5 \in O \Rightarrow 3, 4 \notin O$ as both 3 & 4 conflict with 5. (*)

Claim 3. $O \setminus \{5\}$ is optimal for $[2]$

(*) $\Rightarrow$ $\mathcal{O} \setminus \{5\} \subseteq [2]$

Claim 4: $\mathcal{O} \setminus \{5\}$ is a valid schedule for $[2]$

Pf idea of Claim 3: Assume $\exists$ valid $\mathcal{O}'$ for $[2]$ s.t. $v(\mathcal{O}') > v(\mathcal{O} \setminus \{5\})$

$\mathcal{O}'' = \mathcal{O}' \cup \{5\}$ $\rightarrow$ Claim 5: $\mathcal{O}''$ is a valid schedule for $[5]$

$\rightarrow$ Claim 6: $v(\mathcal{O}'') > v(\mathcal{O})$

$$v(\mathcal{O}'') = v(\mathcal{O}') + v_5 > v(\mathcal{O} \setminus \{5\}) + v_5 = v(\mathcal{O}).$$

$\Rightarrow$ contradicts optimality of $\mathcal{O}$.

Let $OPT(i)$ denote the value of an optimal solution for $[i]$ $\quad 1 \le i \le 5$.

$$OPT(5) = \max\{\underbrace{OPT(4)}_{5 \notin \mathcal{O}},\ \underbrace{OPT(2) + v_5}_{5 \in \mathcal{O}}\}$$

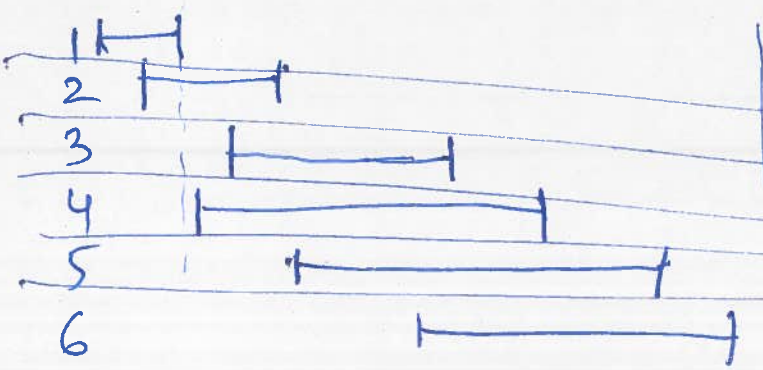
For general n: $\mathcal{O}_j$ be an optimal soln for $[j]$

$$OPT(j) = v(\mathcal{O}_j)$$

Case 1: $n \notin \mathcal{O}_n$ $\quad OPT(n) = OPT(n-1)$

Case 2: $n \in \mathcal{O}_n$ $\rightarrow$ Figure out all j s.t. j conflicts with n.

Def: $p(j)$ be the largest $i < j$ s.t. i & j do not conflict ($p(j) = 0$ is no such i $\exists$)



$p(1) = 0$
$p(2) = 0$
$p(3) = 1$
$p(4) = 1$
$p(5) = 2$
$p(6) = 2$

$\Rightarrow$ All of $p(j)+1, \ldots, j-1$ conflict with j (if $p(j) \neq 0$)

$$OPT(n) = v_n + OPT(p(n))$$

$$OPT(n) = \max\{ OPT(n-1),\ OPT(p(n)) + v_n \}$$

More generally $\forall\ 1 \le j \le n$

$$OPT(j) = \max\{ OPT(j-1),\ OPT(p(j)) + v_j \}$$

$$OPT(0) = 0,\ OPT(1) = v_1.$$