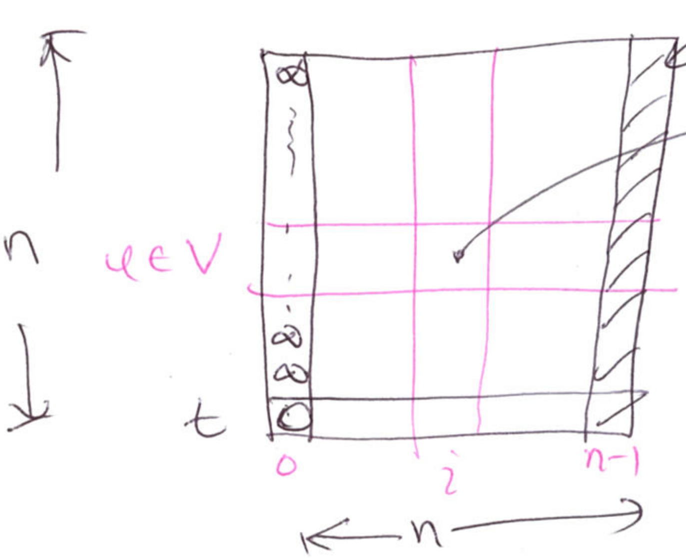
Dec 2

$OPT(u,i)$ = cost of shortest u-t path that uses $\leq i$ edges



$M[u][i] = OPT(u,i)$

[Note: in book uses $OPT(i,u)$]

$\to n^2$ subproblems

Output: $OPT(u, n-1)$

we know ∃ a shortest u-t path that is simple.

Recursive formula

$$OPT(t,0) = 0$$
$$OPT(u,0) = \infty \quad \forall u \neq t$$

$OPT(u,i) \equiv$ $\underbrace{u \to \circ \to \cdots \to t}_{\leq \cancel{t}\, i \text{ edges}}$ [Assume $i \geq 1$]

Case 1: A shortest u-t path $\leq i$ edges actually uses $\leq i-1$ edges.

$$OPT(u,i) = OPT(u,i-1)$$

Case 2: All shortest u-t path $\leq i$ edges use <u>exactly</u> i edges.

$\Rightarrow \exists\, w$ s.t. $(u,w) \in E$ & (u,w) is the first edge in a ~~the~~ shortest u-t path with $= i$ edges.

$$OPT(u,i) = c(u,w) + OPT(w,i-1)$$

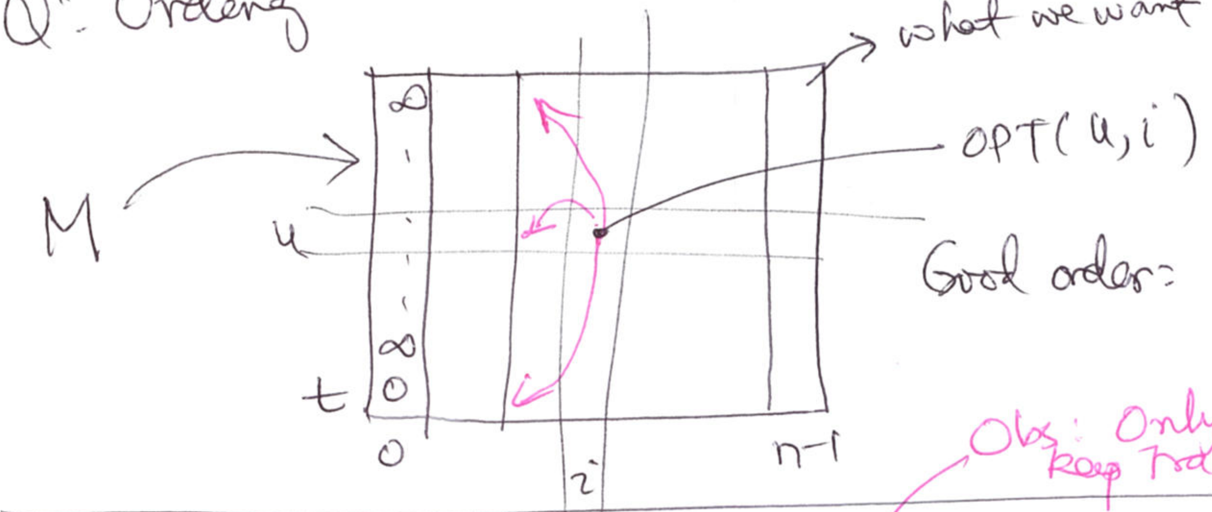
More generally,

$$OPT(u,i) = \min_{(u,w)\in E} c(u,w) + OPT(w,i-1)$$

Overall,

$$\boxed{OPT(u,i) = \min\left\{ OPT(u,i-1),\ \min_{(u,w)\in E}\left\{ c_{(u,w)} + OPT(w,i-1) \right\} \right\}}$$

Q: Ordering



Good order: Column by column.

Obs: Only need to keep track of prev column.

Bellman-Ford

0. $M[t][0] = 0$, $M[u][0] = \infty \quad \forall u \neq t$ $\} O(n)$

Overall $= O(n) + O(n^3) + O(n) = O(n^3)$

1. for $i = 1 \ldots n-1$ $\leftarrow \leq n$

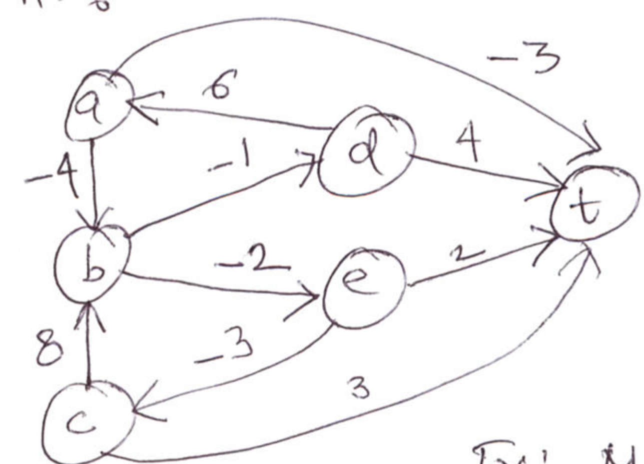
   for $u \in V$ $\leftarrow \leq n$

   $M[u][i] = \min\left\{ M[u][i-1],\ \min_{(u,w)\in E}\left\{ c_{(u,w)} + M[w][i-1] \right\} \right\}$

   $O(n^3)$, $O(n)$ $\leftarrow O(Out_u + 1)$

2. Output $M[s][n-1]$ $\forall s \in V$ $\leftarrow O(n)$

$n = 6$



	0	1	2	3	4	5
a	∞	-3				
b	∞	∞				
c	∞	3	3			
d	∞	4				
e	∞	2				
t	0	0	0	0	0	0

Ex: $M[t][i] = 0 \quad \forall i$

$M[a][1] = \min\{M[a][0], \min\{-4+M[b][0], -3+M[t][0]\}\}$

$= \min\{\infty, \min\{-4+\infty, -3+0\}\} = -3$

$M[b][1] = \min\{\infty, \min\{-1+\infty, -2+\infty\}\} = \infty$

$M[c][1] = \min\{\infty, \min\{8+\infty, 3+0\}\} = 3$

$M[d][1] = \min\{\infty, \min\{6+\infty, 4+0\}\} = 4$

$M[e][1] = \min\{\infty, \min\{-3+\infty, 2+0\}\} = 2$

$M[c][2] = \min\{3, \min\{8+\infty, 3+0\}\} = 3$

Better runtime analysis:

Step 2: $n \cdot \left(\sum_{u \in V} O(out_u + 1)\right) = O\left(n \overbrace{\sum_{u \in V}(out_u+1)}^{= m+n}\right)$

$O(n(m+n)) = O(\cancel{nm+n})$
