Sep 22

Def: An <u>undirected</u> graph $T = (V, E)$ is a tree if (1) T is connected & (2) T has no cycles

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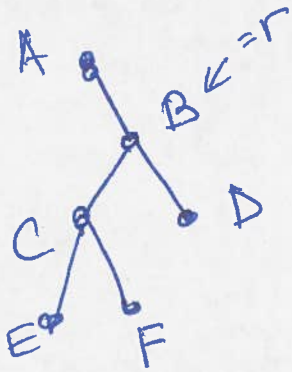
<u>THEOREM 1</u>: A tree on n nodes has <u>exactly</u> $n-1$ edges.

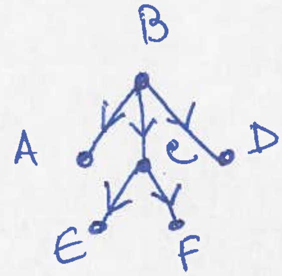


$n = 6, m = 5$
$= n-1$

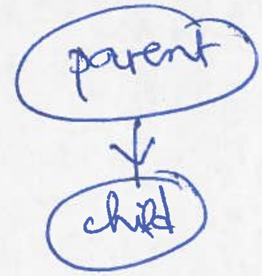
<u>Pf idea:</u> Pick a vertex $r \in V$ & "root" T at r

① Root T at r

② Direct edges "away" from r.



Aside: there are n rooted versions of a tree.



③ Every edge has a unique child

④ r is the <u>only</u> node that is NOT a child of some (directed) edge.

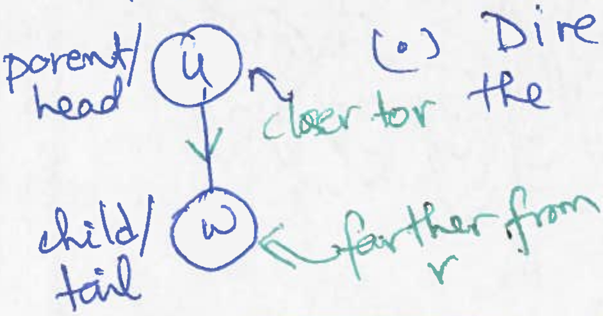
$\Rightarrow |E| = |V \setminus \{r\}| = n-1$

set difference

Pf details: Pick a $r \in V$ & root T at r.

<u>Ex</u>: For every edge $(u, w) \in E$, one of them is closer to r than other.

(.) Direct edges from closer node to the farther node



Ex → <u>Claim 1</u>: r is not the ~~head~~ ~~tail~~ child of any (directed) edge.

Ex → <u>Claim 2</u>: Every (directed) edge has a unique child

Next → Claim 3: Every non-root vertex is the child of ≤ 1 edge

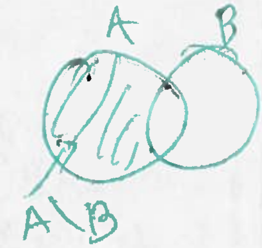
Claim 4: Every non-root vertex is the child of some (directed) edge.

Ex.

Claims 1, 2, 3, 4 $\Rightarrow$ $\exists$ a 1-to-1 correspondence between E & $V \setminus \{r\}$

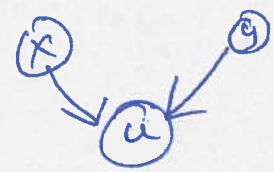
$\Rightarrow$ $m = |E| = |V \setminus \{r\}| = n-1$ ∎

$A \setminus B = \{a \in A \mid a \notin B\}$

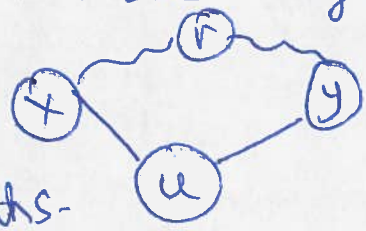


Pf (idea/details) of Claim 3: Proof by contradiction.

For the sake of contradiction, assume vertex u is the child of ~~for~~ two directed edges



Now consider T:



Since T is connected

$\Rightarrow$ $\exists$ r-x & r-y paths.

but note the $u, x \sim r \sim y, u$ is a cycle

$\Rightarrow$ contradicts the fact that T has no cycles! ∎

THEOREM 2: Let T be an undirected graph. Then ANY of the following two properties implies the 3rd:

(1) T is connected

(2) T has no cycles

(3) T has $n-1$ edges.

THEOREM 1: (1) + (2) $\Rightarrow$ (3)

(1) + (3) $\Rightarrow$ (2)

(2) + (3) $\Rightarrow$ (1).