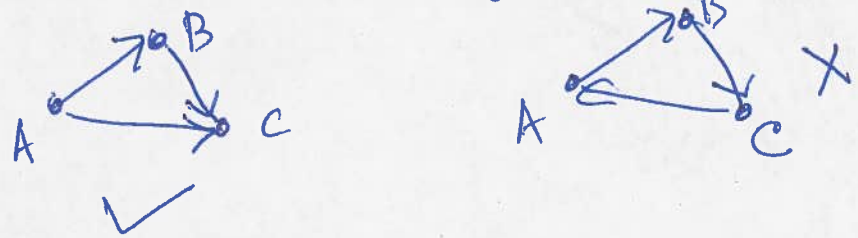
Oct 2

Directed Acyclic Graphs (DAGs)

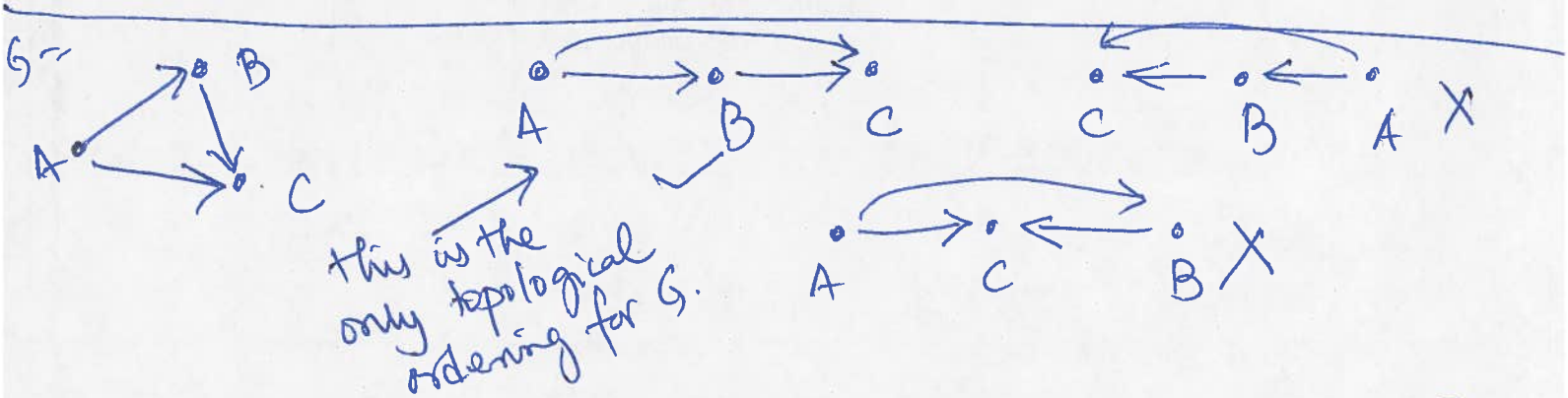
Def: A directed graph G is a DAG if it has no (directed) cycles in it.

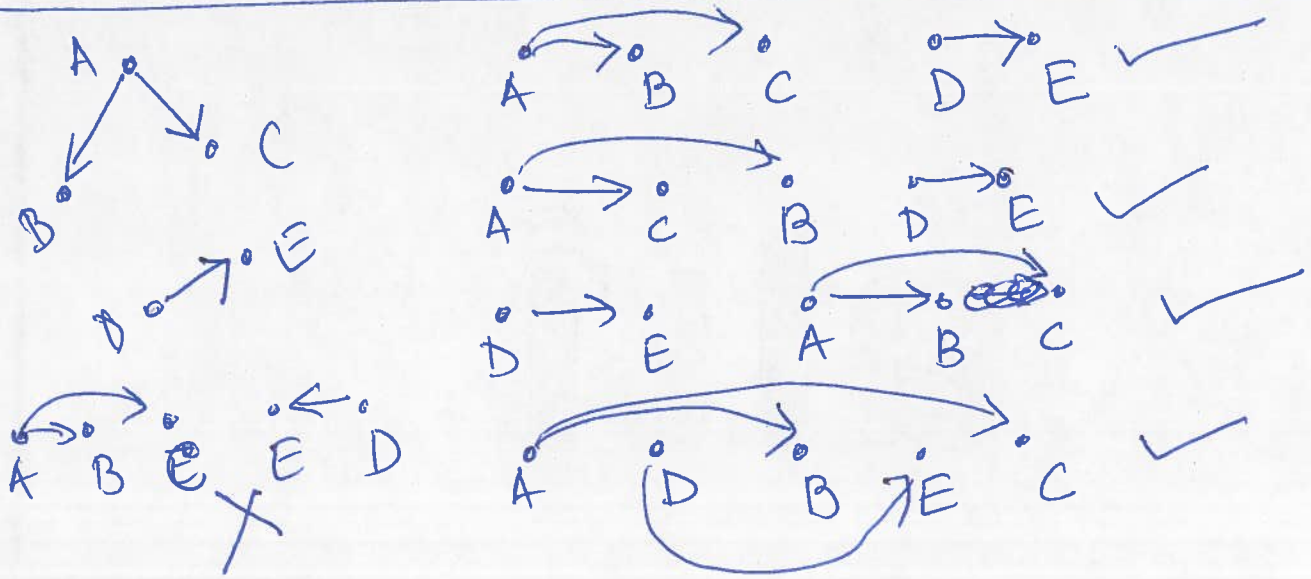


Def: A topological ordering (sorting) of a directed graph $G=(V,E)$ is an ordering of the vertices $u_1, \ldots, u_n$ s.t.

$(u_i, u_j) \in E \Rightarrow i < j$

$[\, u_1 \; u_2 \; \ldots \; u_n \,]$





Problem: **Input:** A directed graph $G=(V,E)$
Output: Topological ordering (if one exists)

Lemma 1: If G has a topological ordering $\Rightarrow$ G is a DAG.

THEOREM: If G is a DAG $\Rightarrow$ G has a topological ordering

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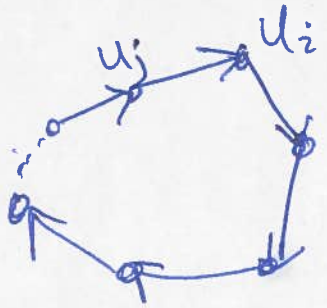
Pf (idea) of Lemma 1: By contradiction ($\neg(A \Rightarrow B)$ equiv $A \wedge \neg B$)

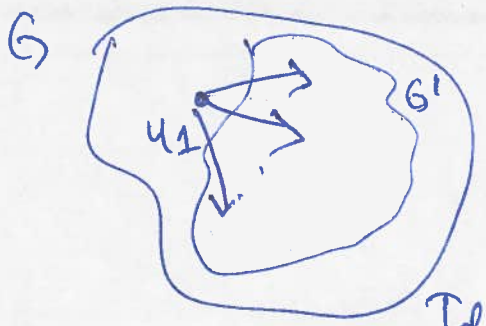
Assume $u_1, .. u_n$ is a topological ordering of G
BUT G has a directed cycle C.



Let i be the smallest index ($\in [n]$) s.t $u_i \in C$
$\Rightarrow$ $\exists\ j$ s.t (u_j, u_i) is as C is a cycle in cycle
$\Rightarrow (u_j, u_i) \in E$
But by choice of i, $j > i$ $\Rightarrow$ $u_1, .., u_n$ is not a topological ordering ■

Pf (idea) of Thm: Via an algo, which given any DAG G, output a topological ordering.

"Reverse engineer" the algo.
Assume: $u_1, .., u_n$ is a topological ordering.
Q: How many incoming edges can u_1 have (in G)?
A: 0



$G' = G \setminus \{u\}$

$\stackrel{def}{=} (V \setminus \{u\}, E \setminus \{(u,w) \in E \mid w \in V$

Lemma 2: G' is a DAG. (Ex)

Idea: Deleting edges cannot create a cycle.

Lemma 3: If G is a DAG, then $\exists$ a vertex u with no incoming edges (On Wed.)

Idea: Recurse! G: DAG.

TopOrd $(G = (V, E))$

1. If $V = \{u\}$, output u
2. Let w be the vertex in G with no incoming edg
   ($\exists$ due to Lemma 3)
3. $G' = G \setminus \{w\}$
4. Output w, TopOrd (G').