Oct 9 $S^* \to$ output of the greedy algo ; $\mathcal{O} \to$ an optimal solution

THM 2: $|S^*| = |\mathcal{O}|$

Notation: $S^* = \{i_1, \ldots, i_k\}$ $\quad$ $\mathcal{O} = \{j_1, \ldots, j_m\}$

$f(i_1) \le f(i_2) \le \cdots \le f(i_k)$ $\quad$ $f(j_1) \le f(j_2) \le \cdots \le f(j_m)$

THM 2': $k = m$.

claim: $k \le m$. [as by defn $\mathcal{O}$ has the max # of intervals in any valid schedule]

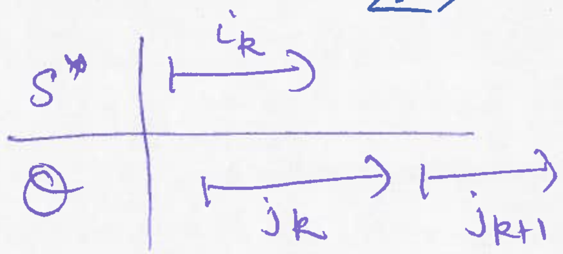
Lemma 1: ("Greedy stays ahead") $\forall\ 1 \le \ell \le k$

$$f(i_\ell) \le f(j_\ell)$$

Pf idea for THM 2' (assume Lemma 1 is true). By contradiction

Assume $k \ne m$. By claim; $k < m$

$\iff m \ge k+1 \Rightarrow j_{k+1} \in \mathcal{O}$



By Lemma 1, $f(i_k) \le f(j_k)$

$(\cdot)$ Consider the time when Greedy adds i_k to S

$\Rightarrow j_{k+1} \in R \Rightarrow R \ne \phi \Rightarrow$ Greedy cannot terminate after adding $i_k \Rightarrow$ contradiction. ∎

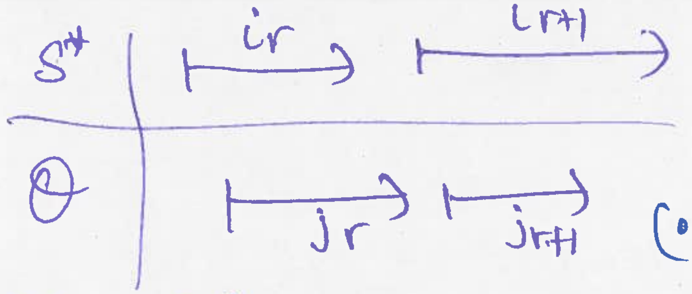
Pf idea of Lemma 1: Proof by Induction on ℓ.

Base case: $f(i_1) \le f(j_1)$ ✓ by definition of algo, i_1 is the job with the earliest finish time [$i_1 = 1$]

Assume: $f(i_\ell) \le f(j_\ell)\ \forall\ 1 \le \ell \le r$ for some $r \ge 1$

Inductive step: Want to show $f(i_{r+1}) \le f(j_{r+1})$

For the sake of contradiction assume $f(i_{r+1}) > f(j_{r+1})$



By I.H. $f(i_r) \le f(j_r)$

(•) Consider the situation after i_r is added to S

(•) Both i_{r+1}, ~~j_{r+1}~~ $j_{r+1} \in R$

Since greedy chose i_{r+1} over j_{r+1} $\Rightarrow$ $f(i_{r+1}) \le f(j_{r+1})$

(a contradiction) ■

Greedy algo (Recall: $f(1) \le \cdots \le f(n)$)

(0) $R = [n]$ $\{O(n)\}$

(1) $S = \phi$ $\{O(1)\}$

(2) While $R \ne \phi$. ⇐ Runs $\le n$ times (b/c step 2.3 removes at least one interval)

$O(n) \to$ (2.1) Pick $i \in R$ w/ smallest index

$O(1) \to$ (2.2) Add i to S

$O(n) \to$ (2.3) Remove all $j \in R$ that conflict with i

(3) Return $S^* = S$ $\{O(n)\}$

OVERALL: $O(n) + O(1) + n \cdot O(n) + O(n)$

$\le O(n^2)$.