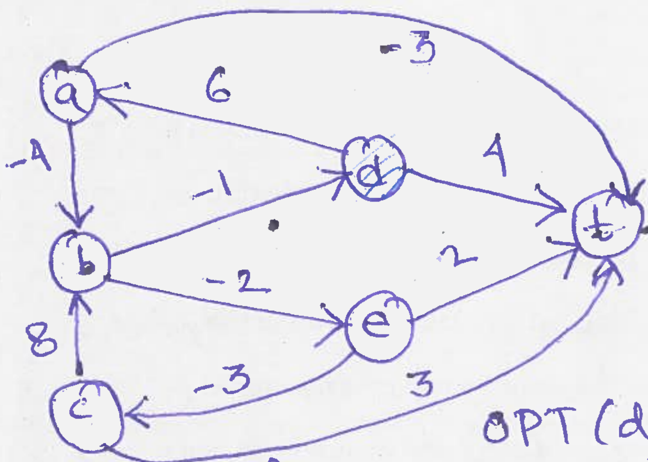
Dec 1

$OPT(u,i)$ = cost of ~~the~~ a shortest u-t path with $\le i$ edges.

<u>Note:</u> i is not explicitly defined parameter of the input.



Let us focus on d:

$OPT(d,0) = \infty$ (as $d \ne t$)

$OPT(d,1) = 4$ (d,t)

$OPT(d,2) = 6-3 = 3$ (d,a,t)

$OPT(d,3) = 3$ $\left\{\begin{array}{l}\cancel{(d,t)} \\ (d,a,t)\end{array}\right.$

$OPT(d,4) = 6-4-2+2 = 2$

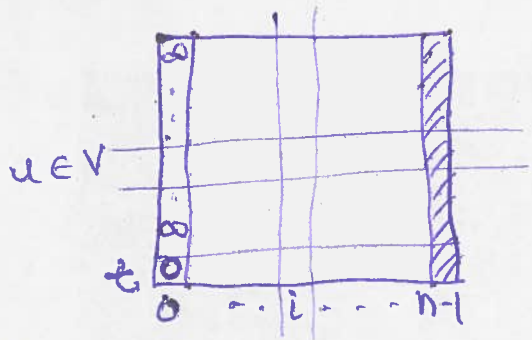
$OPT(d,5) = 6-4-2-3+3 = 0$ (d,a,b,e,t)

(d,a,b,e,c,t)

$\underbrace{OPT(d,6) = 0 = OPT(d,7) = \cdots}$

as there is always a shortest path with $\le n-1$ edges $(=5)$

$OPT(u,i)$ $u \in V$, $0 \le i \le n-1$



$M[u][i] = OPT(u,i)$

$\rightarrow$ # sub-problem $= n^2$

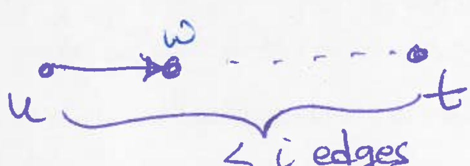
$\rightarrow$ Output: $OPT(s,n-1)$ $\forall s \in V$

<u>Recursive formula:</u>

$OPT(t,0) = 0$

$OPT(u,0) = \infty$ $\forall u \ne t$

$OPT(u,i)$:
$i > 0$



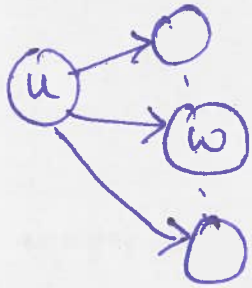
<u>Case 1:</u> A shortest u-t path uses $\le i-1$ edges

$OPT(u,i) = OPT(u,i-1)$.

<u>Case 2:</u> All shortest u-t paths use EXACTLY i edges

$\Rightarrow \exists$ w s.t. $(u,w) \in E$ and a shortest u-t path with i edges uses (u,w) as the first edge

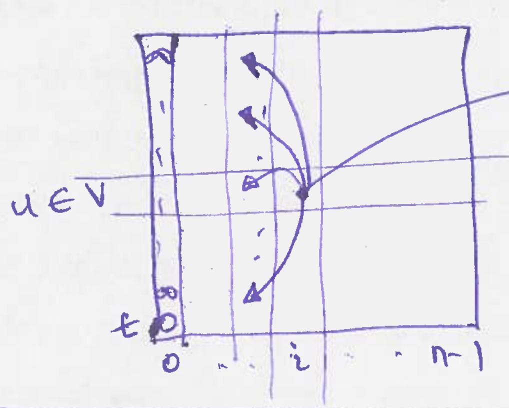
$$OPT(u,i) = c_{(u,w)} + OPT(w, i-1)$$



$$OPT(u,i) = \min_{(u,w)\in E} \{ c_{(u,w)} + OPT(w, i-1) \}$$

$$\boxed{\text{OVERALL: } OPT(u,i) = \min\left\{ OPT(u,i-1), \min_{(u,w)\in E} \{ c_{(u,w)} + OPT(w,i-1) \} \right\}}$$

Ordering?



$OPT(u,i)$: i^{th} column ONLY depends on $(i-1)^{th}$ column

$\Rightarrow$ good ~~order~~ ordering: column by column (left to right)

Bellman-Ford

0. Allocate an $n \times n$ matrix M $\leftarrow O(n^2)$
1. $M[t][0] = 0$, $M[u][0] = \infty$ $\forall u \neq t$ $\leftarrow O(n)$
2. n^2 iterations { for $i = 1 .. n-1$
   for $u \in V$ $\swarrow O(n_u + 1)$
   $O(n) \rightarrow M[u][i] = \min\{ M[u][i-1], \min_{(u,w)\in E} \{ c_{(u,w)} + M[w][i-1]$
3. Output $M[s][n-1]$ $\forall s \in V$

	0	1	2	3	4	5
a	∞	-3				
b	∞					
c	∞					
d	∞					
e	∞					
t	0	0	0	0	0	0

$OPT(t,i) = 0 \ \forall i$

$M[a][1] = \min\{ M[a][0], \min\{-4+\infty, -3+0\}\}$
$= -3$

Overall runtime: $O(n^3)$

Better analysis: $\underset{\#\text{itr of } i}{n} \cdot \left(\sum_{u\in V} O(1+n_u) \right)$
$= O(n(n+m))$