Sep 8

THEOREM: For every input → {M, W, 2n pref list} (instance)
the GS algo outputs a stable matching

⇒ **COROLLARY:** Every instance of the Stable Matching problem has a stable matching.

LEMMA 1: For every i/p, GS algo terminates in $\leq n^2$ iterations

LEMMA 2: The output of the GS algo (S) is a perfect matching

LEMMA 3: S has no instability

LEMMAS 1+2+3 ⇒ prove THEOREM.

Pf of Lemma 1 Pf. idea: Via a progress measure (see support page)

First, note that in every iteration, a new proposal is made
⇒ # iterations = # proposals
proposal $\leq$ # pairs (m, w) $= |M \times W| = \underbrace{|M| \cdot |W|}_{= n^2}$ (*) ∎

Pf-details: At the end of iteration t (≥ 1), let $P(t)$ be the total # of proposals made so far.

(1) $P(1) = 1$ (as in the first iteration, some free woman w proposes to her top man)

(2) $P(t+1) = P(t) + 1$ (as the chosen free woman w proposes to the best man m she has NOT proposed to)

(3) $P(t) \leq n^2$ (by (*)) [(1)+(2)+(3) ⇒ Lemma 1] ∎

$\boxed{\text{OBS 0}}$: S is a matching (induction on # iterations + algo statement)

LEMMA A: If at the end of an iteration, ∃ a free woman w ⇒ w has NOT proposed to all men
[Pigeon-hole principle]

Pf of LEMMA 2:

$\boxed{\text{Pf idea:}}$ By contradiction (use OBS ○, LEMMA 4 & algo definition)