Oct 29

Dijkstra's Algo

$d'(w) = \min_{\substack{u \in R \\ (u,w) \in E}} \{ d(u) + \ell_{(u,w)} \}$

0. $R = \{s\}$, $d(s) = 0$
1. While $\exists\, x \notin R$, $u \in R$ s.t. $(u, x) \in E$

   Pick w that min $d'(w)$

   Add w to R

   $d(w) = d'(w)$

Def: Let P_u be the s-u path in the "Dijkstra tree"

THEOREM: $\forall\, u \in V$, P_u is a shortest s-u path.

$\xrightarrow[\text{(Ex.)}]{}$ $d(u)$ is computed correctly $\Rightarrow$ Dijksta is correct

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Lemma 1: At the end of each iteration of while loop, $\forall\, u \in R$, P_u is a shortest s-u path.

Lemma 2: $u \in V$ s.t $\exists$ s-u path $\Longleftrightarrow$ $u \in R$ at the end of the algo

(Pf: as Ex)

Lemmas 1+2 $\Rightarrow$ THEOREM

Pf (idea) of Lemma 1: By induction on $|R|$

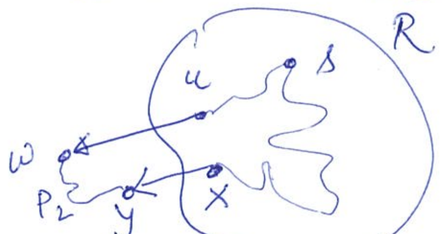
Base case: $|R| = 1 \Rightarrow R = \{s\}$, $d(s) = 0$, P_s: empty path ✓

I.H.: Assume statement is true $|R| = k$ $(k \geq 1)$

I.S.: Argue for $|R| = k+1$

Assume w is added to R at the end of iteration $k+1$

Assume w was "discovered" by u. $\leftarrow$ $d(w) = d(u) + \ell_{(u,w)}$



$P_w = P_u, w$

Claim: P_w is a shortest s-w path.

↳ Argue by contradiction

Assume $\exists$ s-w path P'_w s.t. $\ell(P'_w) < \ell(P_w)$ —— (*)

As $s \in R$, $w \notin R \Rightarrow P'_w$ "crosses" R at some point.

$\Rightarrow \exists\ x \in R,\ y \notin R$ s.t. $(x,y) \in E$

$P'_w = P_x, y, P_2$ (Ex: WLOG assume that s-x path in P'_w is P_x)

$\ell(P'_w) = \ell(P_x) + \ell_{(x,y)} + \ell(P_2)$

I.H. $\rightarrow$ $= d(x) + \ell_{(x,y)} + \underbrace{\ell(P_2)}_{\geq 0 \text{ as } l_e \geq 0\ \forall e \in E}$ since Algo chose w

$\geq d(x) + \ell_{(x,y)} \overset{\text{def of } d'}{\geq} d'(y) \overset{\text{Algo def}}{\geq} d'(w) = d(w) = \ell(P_w)$

~~$\Rightarrow d'(w) \geq d$~~ $\ell(P'_w) \geq \ell(P_w) \Rightarrow$ contradicts (*) ∎