Oct 31

Minimum Spanning Tree (MST)

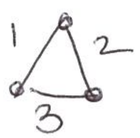
Input: $G = (V, E)$, $c_e \geq 0 \quad \forall e \in E$

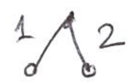
$m \geq n-1 \Leftarrow$ connected, undirected

[≥ 0 for convenience]

Output: (i) $E' \subseteq E$ s.t. $T = (V, E')$ is connected (def: T is a sub-graph)

(ii) minimize $c(T) = \sum_{e \in T} c_e$

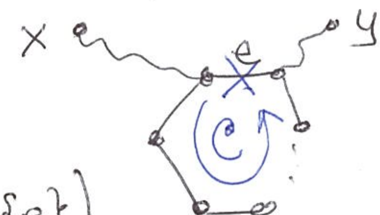




PROPOSITION: Let $c_e > 0 \; \forall e \in E$. Then any optimal solution T is a tree

Pf (idea) By contradiction: Assume T is not a tree

$\Longrightarrow$ as T is connected, T has a cycle C. Fix any $e \in C$



Delete e from T : $T' = (V, E' \setminus \{e\})$

Claim 1: T' is connected

Claim 2: $c(T') < c(T)$

Claim 1 + 2 $\Rightarrow$ contradicts the fact that T is optimal.

Pf idea for Claim 1: Consider any $x, y \in V$

→ $\exists$ x-y path in T without e $\Rightarrow$ x, y still connected in T'.

→ All x-y paths in T use e → use rest of the cycle

$\Rightarrow$ x, y are connected in T'

Pf of claim 2: $c(T') = c(T) - c_e < c(T)$ ∎

Brute force ~~alg~~ algo: Go over all possible $E' \subseteq E$ ← 2^m many E'

(i) check if (V, E') is connected ← $O(m+n) = O(m)$

(ii) Maintain min cost over all such E' $\Rightarrow O(m \cdot 2^m)$