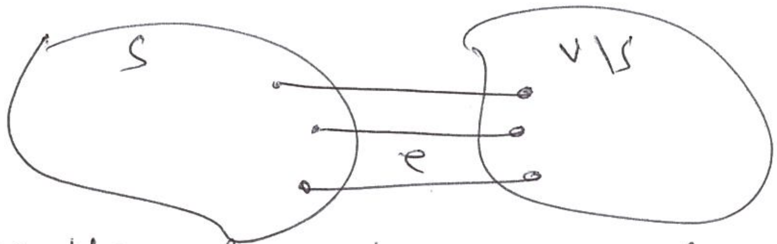
Nov 2

Cut Property Lemma

ASSUME: All c_e all distinct

For all cuts $(S, V \setminus S)$ $S \neq \phi$, $V \setminus S \neq \phi$



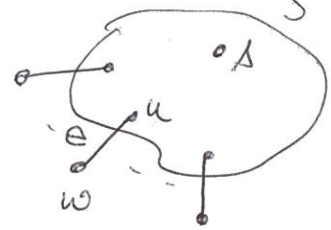
Let e be the cheapest crossing edge

$\Rightarrow$ e is in ALL MSTs.

Assume Cut property Lemma is true (all c_e's are distinct)

Thm 1: Prim's algo is correct.

Pf (Idea): Consider the run of the algo where it is about to add an edge e to T



Goal: show that e is the cheapest crossing edge for some cut $(S, V \setminus S)$

Idea: Apply the cut property lemma to S from algo.

Claim 1: $S \neq \phi$ (as $u \in S$)

Claim 2: $S \neq V$ $(w \notin S)$

by algo design

Claim 3: e is the cheapest crossing edge $(S, V \setminus S)$

$\Longrightarrow$ adding e was a "safe" choice.

Cut property Lemma

Pf: EX

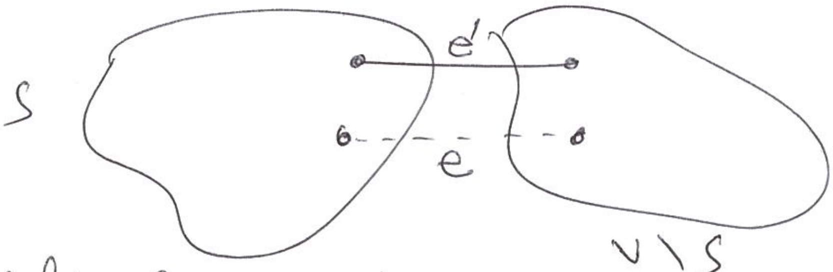
Claim 4: At the end of each iteration (S, T) is connected

$\Longrightarrow$ (V, T) is connected. (since at the end $S = V$)

at the end

Pf (idea) of cut property lemma   By contradiction

Assume $\exists S$ s.t $S \neq \Phi$, $S \neq V$ & an MST T s.t the cheapest crossing edge $(S, V \setminus S)$ is NOT in T.



As $(V, \cancel{E})$ is connected
T
$\Rightarrow \exists$ a crossing edge e' $(S, V \setminus S)$ s.t $e' \in T$

Goal: Create another spanning tree T' s.t $c(T') < c(T)$

→ Consider: $T' = (T \setminus \{e'\}) \cup \{e\}$

$c(T') = c(T) - c_{e'} + c_e$

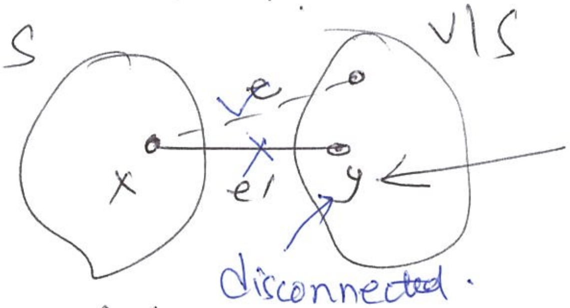
$< c(T)$?

Claim: $c_{e'} > c_e$

(1) e is the cheapest crossing edge $\Rightarrow c_{e'} \geq c_e$

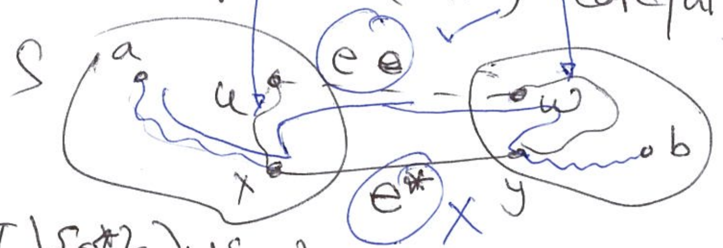
(2) All edge costs + distinct

Issue: Dropping e' could disconnect T'.



as T is connected

fix: we'll drop a (more) carefully chosen edge.



As T is connected
$\Rightarrow \exists$ a u-w path
as $u \in S$ & $w \notin S$
$\Rightarrow \exists$ a crossing edge $e^* = (x, y)$ on this path.

$T' = (T \setminus \{e^*\}) \cup \{e\}$

As before: $c(T') < c(T)$

Claim: T' is connected

Case 1: a-b path does NOT use e^* ✓

Case 2: a-b path has e^* then take the "scenic" path in T'.

∎

THM 2: Kruskal's is correct (consider edges in increasing order of c_e & add e if adding it does not create a cycle)

Pf (dea): Consider the case when $e = (u, w)$ is being added to T.

Goal: e is the cheapest crossing edge for some cut $(S, V \setminus S)$

Q: What is S?

A: Let S be set of vertices connected to u via edges in T