Sep 9 THEOREM: For every input (M, W, $2n$ pref list) the GS algo outputs a stable matching.

Lemma 1: For every input, GS terminates in $\leq n^2$ iterations

Lemma 2: The output of GS algo (S) is a perfect matching

Lemma 3: S has no instability.

Lemmas $1+2+3 \Rightarrow$ THEOREM.

Pf idea of Lemma 1: In each iteration, a new proposal (from w to m) is made.

$\Rightarrow$ # iterations $=$ # proposals $\leq$ # pairs $(w, m) \overset{=}{\leq} |W \times M| = n^2$ ∎

Obs 0: S is a matching

Obs 1: Once a man gets engaged, he keeps on getting engaged to better women.

Obs 2: If w proposes to m after $m' \Rightarrow m' > m$ in L_w

Lemma 4: If at end of an iteration, w is free $\Rightarrow$ w has NOT proposed to all men.

Pf of Lemma 2:

Pf idea: Pf by contradiction (Use Obs 0, Lemma 4, Lemma 1, algo definition)

Pf details: Assume S is NOT a perfect matching.

$\Rightarrow$ $\exists$ a free woman w

(by Obs 0 + algo def) $\underset{\text{(Lemma 4)}}{\Rightarrow}$ $\exists$ a man m that w has not proposed to ——— $(*)$

Since algo has terminated (Lemma 1) $\underset{\text{(algo def.)}}{\Rightarrow}$ all free women have proposed to all men

$\Rightarrow$ contadicts $(*)$ ∎

Pigeon-hole principle: If $\leq n-1$ pigeons are put in n holes $\Rightarrow \exists$ atleast one empty hole.

Pf of Lemma4! Pf idea! Pf by contradiction (Pigeon-hole principle + obs 1 + Algo def/code)

Pf details! Assume $\exists$ a free woman w who has proposed to all men

$\xRightarrow{\text{Obs 1 + algo def}}$ all men m are engaged (@)

Since w is free $\Rightarrow \leq n-1$ women are engaged

$\xRightarrow{\text{(PHP)}}$ $\leq n-1$ engaged men $\Rightarrow$ contradicts (@)

(hole :: men
pigeon :: women
assignment :: engaged)