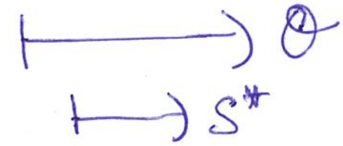
Oct 2

Correctness of greedy algo

THM 1: S^* is an optimal schedule

Let $\mathcal{O}$ be an optimal solution

(Wrong) Idea: Try to prove $S^* = \mathcal{O}$



THM 2: $|S^*| = |\mathcal{O}|$

Notation: $S^* = \{ i_1, \dots, i_k \}$ $\quad f(i_1) \le f(i_2) \le \dots \le f(i_k)$

$\mathcal{O} = \{ j_1, \dots, j_m \}$ $\quad f(j_1) \le f(j_2) \le \dots \le f(j_m)$

$\boxed{\text{THM } 2': \quad k = m}$

Claim 1: $k \le m$ (as $\mathcal{O}$ is optimal)

Lemma 1 ("Greedy stays ahead") $\forall \; 1 \le l \le k$

$$f(i_l) \le f(j_l)$$

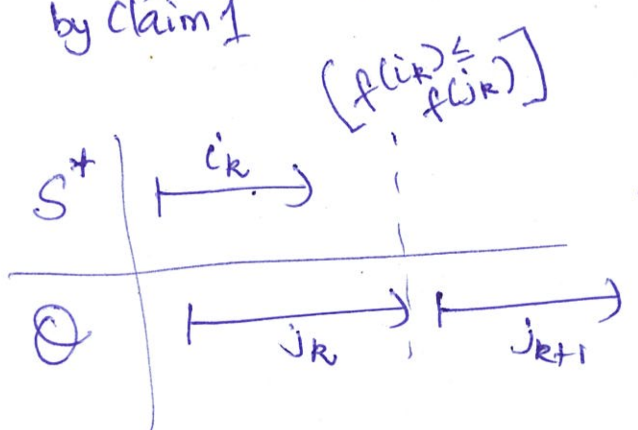
(Assume Lemma 1 is true)

Pf (idea) of THM 2': By contradiction: $k \ne m$

$\xRightarrow{\text{by Claim 1}}$ $k < m \iff m \ge k+1$

$\Rightarrow j_{k+1} \in \mathcal{O}$

By Lemma 1, $f(i_k) \le f(j_k)$



→ Consider the situation after i_k is added to S

$\Rightarrow j_{k+1} \in R$ (as j_{k+1} does not conflict with any i_l $l \le k$)

Ex

$\Rightarrow R$ is non-empty $\Rightarrow$ Greedy algorithm did not terminate $\Rightarrow$ contradiction! ∎

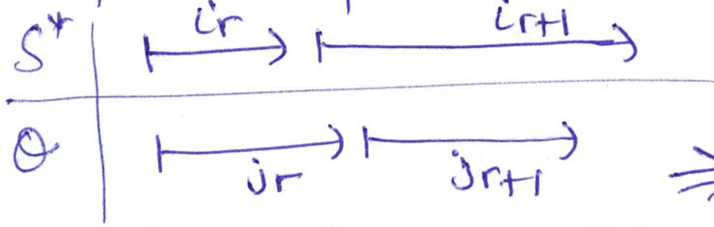
Pf (idea) of Lemma 1: By induction on ℓ

Base case: $\ell = 1$. $f(i_1) \le f(j_1)$ ← By algo defn, $f(i_1)$ is the smallest finish time

I.H.: $f(i_\ell) \le f(j_\ell)$ $\forall\ 1 \le \ell \le r$ $(r \ge 1)$

I.S.: show $f(i_{r+1}) \le f(j_{r+1})$

for sake of contradiction assume $f(i_{r+1}) > f(j_{r+1})$



(•) Consider the case after i_r is added to S by greedy

$\Rightarrow$ $i_{r+1} \in R$ $\Rightarrow$ i_{r+1} cannot be picked by greedy

$j_{r+1} \in R$ ← Ex!

Contradiction! ← (as $f(j_{r+1}) < f(i_{r+1})$

Recall: $f(1) \le \ldots \le f(n)$

Greedy algo:

$O(n)$ {
0. $R = [n]$ ← $O(n)$
1. $S = \phi$ ← $O(1)$
}

2. While $R \ne \phi$ ← $\le n$

$O(n)$ {
(2.1) Pick $i \in R$ with smallest index ← $O(n)$
(2.2) Add i to S ← $O(1)$
(2.3) Remove all j that conflict with i from R. ← $O(n)$
}

3. Return $S^* = S$ ← $O(n)$

Overall runtime: $O(n) + n \cdot O(n) + O(n) = O(n) + O(n^2) + O(n)$
$= O(n^2)$ ∎