Oct 9

Lemma 1: At the end of each iteration, if $u \in R$, the P_u is a shortest s-u path.

Pf (idea): By induction on $|R|$

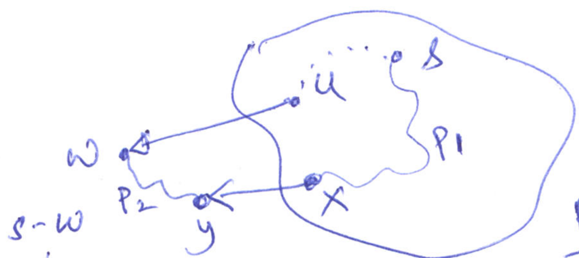
Base case: $|R| = 1$, $R = \{s\}$, $d(s) = 0$ ✓

I.H.: Assume lemma is true $|R| = k$ $(k \geq 1)$

I.S.: Argue $|R| = k+1$

Assume w is the $(k+1)^{th}$ vertex added to R

Assume w was "discovered" from u $(\equiv d(w) = d(u) + \ell_{(u,w)})$



$P_w = P_u, w$

Goal: Argue P_w is a shortest s-w path.

Pf: By contradiction.

$\exists$ a s-w path P'_w s.t. $\ell(P'_w) < \ell(P_w)$ —— (*)

As $s \in R$ and $w \notin R \Rightarrow P'_w$ "crosses" R at some point

$\Rightarrow \exists\, x \in R,\ y \notin R$ s.t. $(x,y) \in E$.

$P'_w = P_1, x, y, P_2$

$\ell(P'_w) = \ell(P_1) + \ell_{(x,y)} + \ell(P_2)$ —— ①

~~$\geq d(x) + \ell_x$~~

defn of $d(x)$ $\rightarrow$ $\geq \underbrace{d(x) + \ell_{(x,y)}}_{\geq d'(y)} + \ell(P_2)$ —— ②

def of $d'(\cdot)$ $\rightarrow$ $\geq d'(y) + \ell(P_2)$ —— ③

as $\ell(P_2) \geq 0$ $\rightarrow$ $\geq d'(y) \geq d'(w) = d(w) = \ell(P_w)$

since w was chosen over y — Algo defn

$\Rightarrow \ell(P'_w) \geq \ell(P_w)$ contradicts (*)! ∎

Minimum Spanning Tree (MST)

Input: $G = (V, E)$ ← undirected, connected , $c_e \geq 0 \quad \forall e \in E$
[↑ for convenience only]

Output: (i) $E' \subseteq E$ s.t. $T = (V, E')$ is connected
↖ (sub-graph of G)

(ii) $\min \; c(T) = \sum_{e \in E'} c_e$

Prop: Let $c_e > 0 \; \forall e \in E$. Then the optimal solution T is a tree.

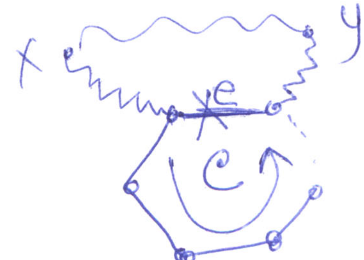
Pf (idea): By contradiction.

Assume T is NOT a tree

$\Longrightarrow$ (as T is connected + undirected) $\exists$ a cycle in T.

Fix any cycle C in T.

Fix any edge $e \in C$.



. Delete e from T to get $T' = (V, E' \setminus \{e\})$

Claim 1: T' is connected, Claim 2: $c(T') < c(T)$

$\Rightarrow$ Claim 1+2 $\Rightarrow$ contradiction (optimality of T).

Pf of Claim 2: $c(T') = c(T) - c_e < c(T)$ as $c_e > 0$.

Pf of Claim 1: Consider $x, y \in V$

Case 1: $\exists$ an x-y path that does not use $e \Rightarrow x, y$ connected in T'!

Case 2: All x-y path use edge $e \Rightarrow$ use the rest of C instead of e

(x, y) connected in T'. $\Longleftarrow$