Oct 11

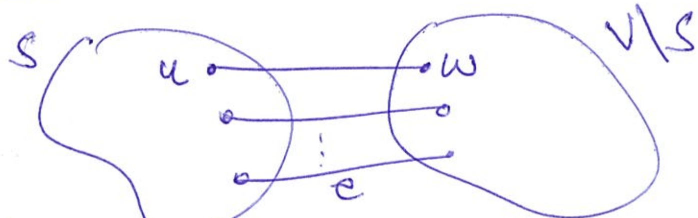
Cut Property Lemma

$S \neq V$

$\Updownarrow$

ASSUME: All c_e's are distinct

For all cuts $(S, V \setminus S)$ s.t. $S \neq \phi$, $V \setminus S \neq \phi$



Consider all "crossing edges"

Let e be the cheapest crossing edge $\Rightarrow$ e is in ALL MSTs for G.

Assume Cut property lemma is true (c_e's are distinct)

THM1: Prim's algo is correct

Pf (idea) Consider the run of the algo when it is about to add e to T.

Goal: e is the cheapest crossing edge for some cut $(S, V \setminus S)$ $\xRightarrow{\text{cut property lemma}}$ this is a "safe" choice

Apply the cut property lemma on $(S, V \setminus S)$ where S is from Prim's algo.

Claim 1: $S \neq \phi$ $(w \notin S)$

Claim 2: $S \neq V$ $(u \in S)$

Claim 3: e is the cheapest crossing edge (follows from algo statement)

$\Longrightarrow$ every edge added by Prim's is correct/safe.

Claim 4: At the end of each iteration (S, T) is connected

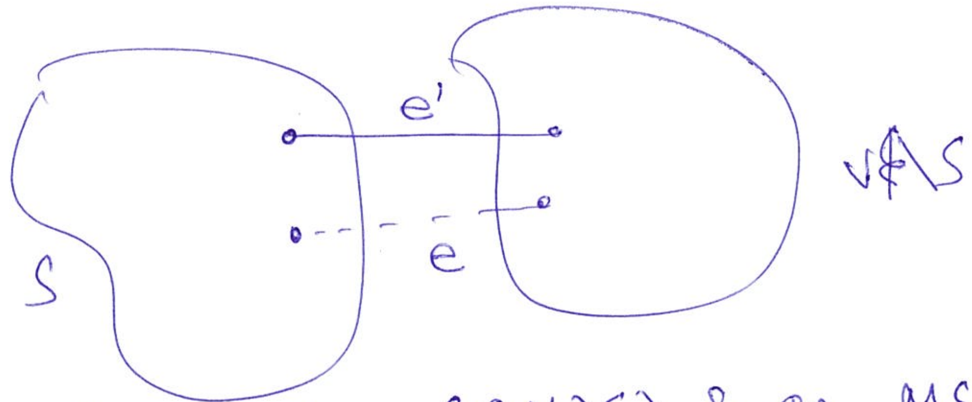
$\Longrightarrow$ At the end of the algo (V, T)

$\boxed{\text{Pf: Ex}}$

Claims 1+2+3+4 $\Rightarrow$ THM 1 ∎

Pf (idea) for ~~Cut~~ cut property lemma:

By contradiction.



Assume $\exists$ a cut $(S, V \setminus S)$ & an MST T s.t. the cheapest crossing edge e is ~~in~~ NOT in T.

Since T is connected, $\exists$ a crossing edge e' in T

Compute / design / look at: $T' = (T \setminus \{e'\}) \cup \{e\}$

$c(T') = c(T) - c_{e'} + c_e$ **Obs:** $c_{e'} > c_e$

$\overset{?}{<} c(T)$ ■