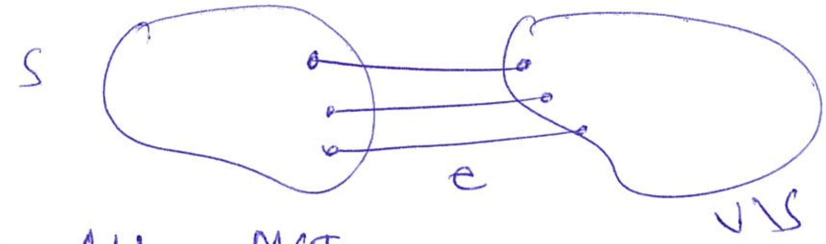
Oct 18

Cut Property Lemma (all c_e's are distinct)

$\forall$ cuts $(S, V\setminus S)$ $\quad \phi \neq S \neq V$

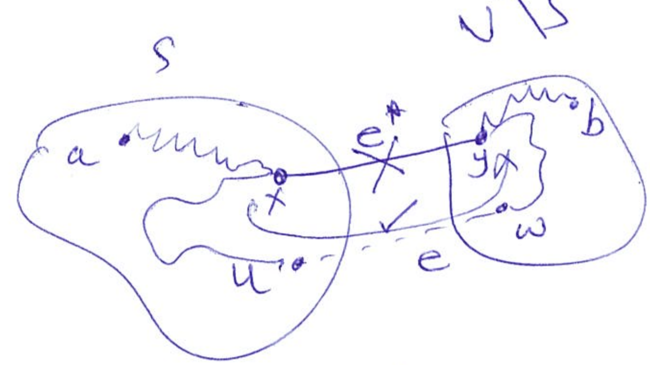


Let e be the cheapest crossing edge

$\Rightarrow e$ is in ALL MSTs.

Pf (idea): By contradiction

Assume not $\Rightarrow \exists$ a cut $(S, V\setminus S)$ s.t. e is the cheapest crossing edge.



$\exists$ an MST T ~~and~~ s.t. $e \notin T$.

Since T is connected $\Rightarrow \exists$ u-w path in T

$u \in S$, $w \notin S$ $\Rightarrow \exists$ $x \in S$, $y \notin S$ s.t. $e^* = (x,y)$ is an edge in T.

Define: $T' = (T \setminus \{e^*\}) \cup \{e\}$

Claim 1: $c(T') = c(T) - c_{e^*} + c_e < c(T)$ as $c_{e^*} > c_e$

Claim 2: T' is connected

Case 1: a-b path doesn't use $e^* \Rightarrow$ ✓

Case 2: a-b ___ does use $e^* \Rightarrow$ take "scenic route" ✓

Claims 1 + 2 $\Rightarrow$ T is NOT an MST $\Rightarrow$ contradiction ∎

THM: Kruskal's algo is correct (consider all edges in increasing order of c_e & add e if adding it does NOT introduce a cycle)

Pf (idea) Consider the case when $e = (u, w)$ is being added to T.

Goal: Show e is the cheapest crossing edge for some cut $(S, V\setminus S)$

Q: What is S?

A: Let S be set of vertices connected to u using ONLY edges in T so far.

Perturbation Trick

Assume: Assume all c_e's are integers (Ex: without this assumption)

Idea: Add to i^{th} edge add an extra $\frac{i}{2mn}$ $1 \le i \le m$

$$c'_{e_i} = c_{e_i} + \frac{i}{2mn}$$

Ex: All c'_e are distincts.

Q: By how much does the MST cost change?

edges in tree T is $= n-1$

$\Rightarrow$ max change in $c(T) \le (n-1) \cdot \frac{m}{2mn}$

$= \frac{n-1}{2n} < \frac{n}{2n} = \frac{1}{2}$.

$\Rightarrow$ cannot "confuse" 2 spanning tree of diff cost since if $c(T_1) \neq c(T_2) \Rightarrow |c(T_1) - c(T_2)| \ge 1$ as all c_e as integers.