Lemma: $T(n) \leq cn\log_2 n + cn$

$\Rightarrow$ MergeSort runs in time $O(n \log n)$

Some remarks:

1. $O(n\log n)$ best known upper bound for general algo.
2. Can do faster if domain of the a_i's is of size $O(n)$ (T/F on piazza $a_i \in \{0,1\}$)
3. Can have faster runtime for "almost" sorted input.
4. Any comparison based ~~alg~~ algo for sorting takes $\Omega(n\log n)$ comparisons. $\circ$

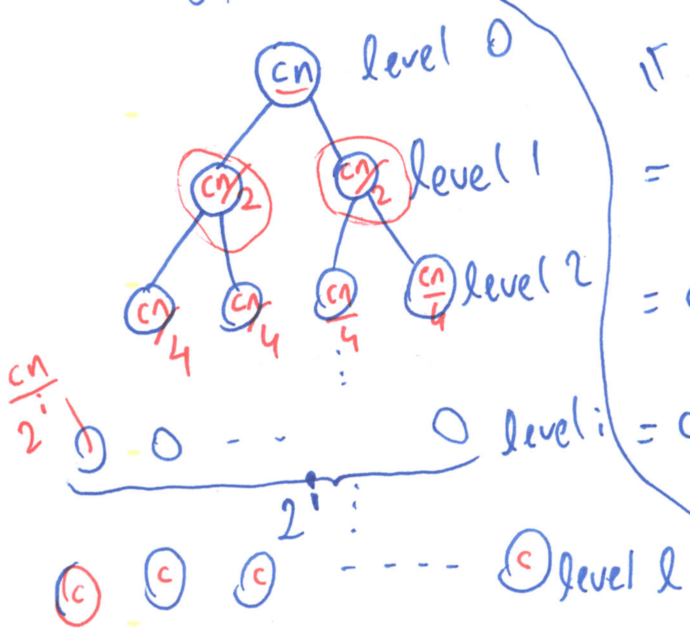
Oct 23

Strategies for solving recurrences

1. "Unroll" the recurrence and use the pattern
2. Guess the answer and verify using induction on n

Pf of Lemma:

Strategy 1:



$$T(n) \leq cn + 2T\left(\frac{n}{2}\right)$$
$$\leq cn + 2\left(\frac{cn}{2} + 2T\left(\frac{n}{4}\right)\right)$$
$$= cn + cn + 4\cdot T\left(\frac{n}{4}\right)$$
$$= cn + cn + 4\left(\frac{cn}{4} + 2T\left(\frac{n}{8}\right)\right)$$
$$= cn + cn + cn + 8\,T\left(\frac{n}{8}\right)$$

Contribution from level $i = 2^i \cdot \frac{cn}{2^i} = cn$

$$\Rightarrow T(n) \leq cn\cdot(\#\text{ levels})$$
$$= cn\cdot(l+1)$$

Note: $\frac{n}{2^{\ell}} = 1 \Rightarrow 2^{\ell} = n$

$\Rightarrow \ell = \log_2 n$

$$\Rightarrow T(n) \leq cn.(\log_2 n + 1)$$

$$= cn.\log_2 n + cn \; \square$$

Strategy 2: **Guess:** $T(n) \leq cn\log_2 n + cn$

$$\Rightarrow T(1) = c.1.\log_2 1 + c$$

$$= c \checkmark$$

Inductive hypothesis:

$$T\left(\frac{n}{2}\right) \leq c\frac{n}{2}\log_2\left(\frac{n}{2}\right) + \frac{cn}{2}$$

$$= \frac{cn}{2}\left(\log_2\left(\frac{n}{2}\right) + 1\right)$$

$$= \frac{cn}{2}\left(\log_2 n - \log_2 2 + 1\right)$$

$$= \frac{cn}{2}.\log_2 n \quad (*)$$

I.S.

By recursion:

$$T(n) \leq cn + 2T\left(\frac{n}{2}\right)$$

by (*) $\leq cn + 2\left(\frac{cn}{2}.\log_2 n\right)$

$$= cn + cn.\log_2 n$$

Assume $T(n) \leq kn + cn \log_b n$

$n=1 \rightarrow T(1) \leq k + c.\log_b 1$

if $b \geq 1$, $b \in N$ $\Rightarrow = k + c.0$

$= k$

Inductive hypothesis $T\left(\frac{n}{2}\right) \leq \frac{kn}{2} + c.\frac{n}{2}.\log_b\left(\frac{n}{2}\right)$

$$\leq \frac{kn}{2} + \frac{cn}{2}(\log_b n - \log_b 2)$$

$\boxed{b=2} \Rightarrow \leq \frac{kn}{2} + \frac{cn}{2}(\log_2 n - 1)$

$$= \frac{kn}{2} + \frac{cn}{2}.\log_2 n - \frac{cn}{2}$$

$$\cancel{= \frac{kn - cn}{2} + c}$$

$$T\left(\frac{n}{2}\right) = \frac{n}{2}(k + c.\log_2 n - c)$$

$$T(n) \leq cn + 2T\left(\frac{n}{2}\right)$$