Nov 15

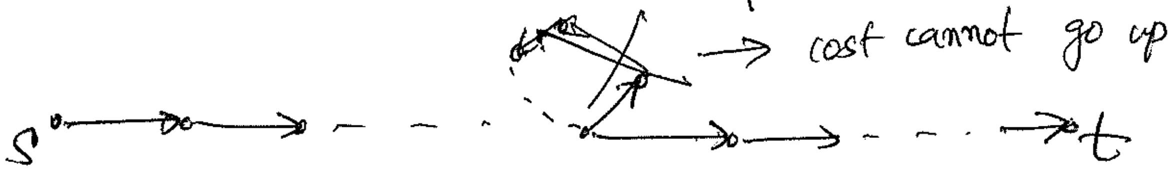
Attempt 5: Bellman-Ford Algo

$OPT(s, i)$ = cost of a shortest s-t path with $\le i$ edges.

$s \in V$, $i \ge 0$ ~~pos~~ integer

Prop: ~~If~~ G has no -ve cycle $\Rightarrow$ $\forall s$ $\exists$ a shortest s-t path that is simple.

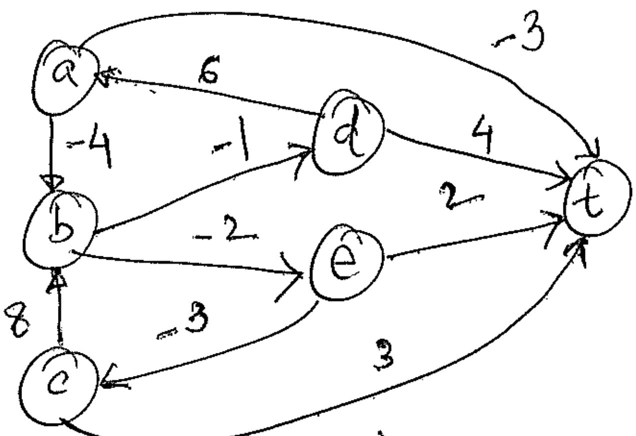
Pf (idea):



$OPT(s, i)$: $0 \le i \le n-1$

$\Rightarrow$ $OPT(s, n-1)$ = cost of a shortest s-t path.

Goal: Compute $OPT(s, n-1)$ $\forall s \in V$



Look at d. (as $d \ne t$)

$OPT(d, 0) = \infty$

$OPT(d, 1) = 4$ [d, t]

$OPT(d, 2) = 6 - 3 = 3$ [d, a, t]

$OPT(d, 3) = 3$ [d, a, t]

$OPT(d, 4) = 6 - 4 - 2 + 2 = 2$ [d, a, b, e, t]

$OPT(d, 5) = 6 - 4 - 2 - 3 + 3 = 0$ [d, a, b, e, c, t]

$OPT(d, 6) = 0$

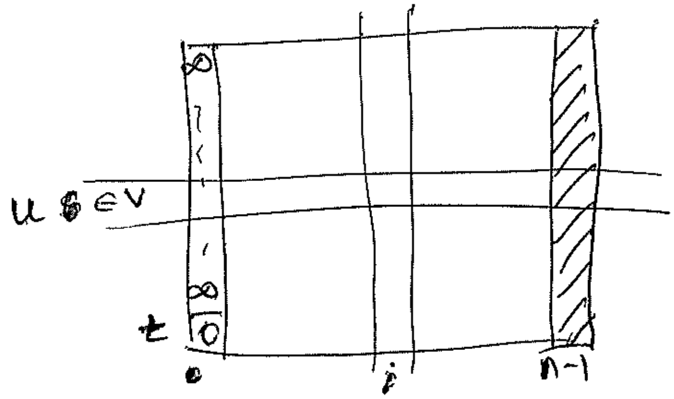
$OPT(d, 7) = 0$

$OPT(d, 6) = OPT(d, 5) = OPT(d, 7) \cdots$

By Prop $n = 6 \Rightarrow OPT(d, 5)$ is cost of a shortest d-t path.

$OPT(s,i)$ = cost of shortest s-t path with $\leq i$ edges

$s \in V$, $0 \leq i \leq n-1$



<u>Goal:</u> $M[u,i] = OPT(u,i)$

subproblems $= n^2$

↑ poly many subproblems ✓ £

→ <u>Output:</u> ~~OPT~~ $M[u, n-1]$ $\forall u \in V$

<u>Recurrence:</u> $OPT(t,0) = 0$

$OPT(u,0) = \infty \quad \forall u \neq t$

$OPT(u,i)$ for $i > 0$

<u>Case 1:</u> $\exists$ a shortest u-t path with $\leq i$ edges that actually uses $\leq i-1$ edges:

$$OPT(u,i) = OPT(u,i-1)$$

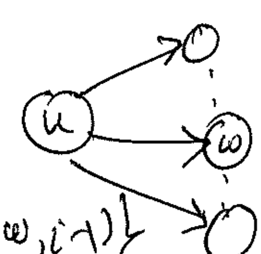
<u>Case 2:</u> All shortest u-t paths with $\leq i$ edges uses EXACTLY i edges.

$i-1$ edges

$\Rightarrow$ shortest w-t path with $\leq i-1$ edges

Know: 1st edge is (u,w)

$$OPT(u,i) = c_{u,w} + OPT(w,i-1)$$



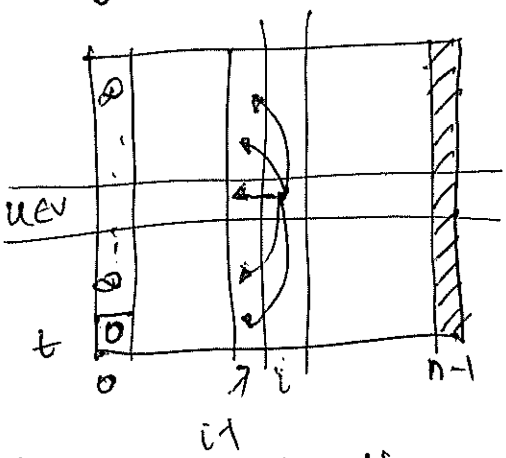
<u>Generally,</u> 1st edge has to be from u to one of its outgoing neighbors

$$OPT(u,i) = \min_{w: (u,w) \in E} \{ c_{u,w} + OPT(w,i-1) \}$$

OVERALL:

$$OPT(u,i) = \min \left\{ OPT(u,i-1), \min_{w: (u,w) \in E} \{ c_{u,w} + OPT(w,i-1) \} \right\}$$

Ordering



$\Rightarrow$ i^{th} column only depends on Column $i-1$

$\Rightarrow$ good ordering: column by column (L to R $0, 1, \dots, n-1$)

Bellman-Ford Algo:

0. Allocate an $n \times n$ matrix M
1. $M[t,0] = 0$, $M[u,0] = \infty \;\; \forall u \neq t$
2. for $i = 1 \dots n-1$
   for $u \in V$
   $$M[u,i] = \min \left\{ M[u,i-1],\ \min_{\substack{w:\\(u,w)\in E}} \{ c_{u,w} + M[w,i-1] \} \right\}$$
3. Return $M[s, n-1] \;\; \forall s \in V$.

	0	1	2	3	4	5
a	∞	-3				
b	∞					
c	∞					
d	∞					
e	∞					
t	0					

$$M[a,1] = \min \{ M[a,0],\ \min \{ -4 + M[b,0],\ -3, M[t,0] \} \}$$
$$= \min \{ \infty, \min \{ -4 + \infty, -3 + 0 \} \}$$
$$= -3$$