Oct 6

Correctness of Greedy algo (Interval Scheduling problem)

Recall: $f(1) \le f(2) \le \cdots \le f(n)$

0. $R \leftarrow [n]$
1. $S \leftarrow \phi$
2. While $R \neq \phi$
   - (2.1) Pick $i \in R$ with smallest index
   - (2.2) Add i to S
   - (2.3) Remove <u>all</u> $j \in R$ s.t. j conflicts w/ i
3. return $S^* \leftarrow S$

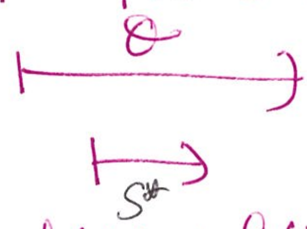
<u>THM 1:</u> For all inputs, S^* is an optimal schedule

i.e. has max # intervals among all valid schedules for the input.

Let $\mathcal{O}$ be an <u>optimal</u> solution

<u>Idea:</u> Prove $S^* = \mathcal{O}$ ✗ as multiple optimal solutions possible

<u>THM 2:</u> $|S^*| = |\mathcal{O}|$



<u>Notation:</u> $S^* = \{i_1, \ldots, i_k\}$ $\quad f(i_1) \le f(i_2) \le \cdots \le f(i_k)$

$\mathcal{O} = \{j_1, \ldots, j_m\}$ $\quad f(j_1) \le f(j_2) \le \cdots \le f(j_m)$

<u>THM 2':</u> $k = m$

<u>Claim 1:</u> $k \le m$ (as $\mathcal{O}$ is an optimal solution)

<u>Lemma 1</u> (Greedy stays ahead) $\forall \; 1 \le \ell \le k$

$$f(i_\ell) \le f(j_\ell)$$

[Assume Lemma 1 is true]

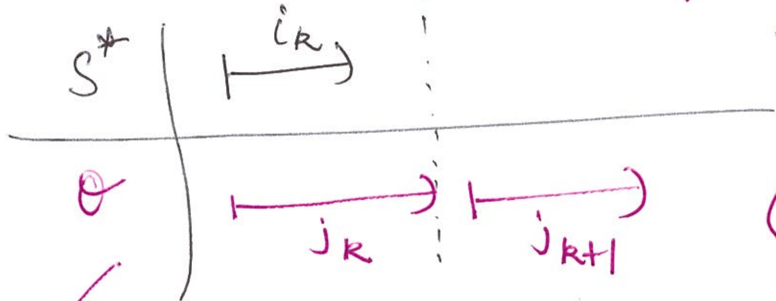
Pf (idea) of THM 2': By contradiction.

Assume $k \neq m \Longrightarrow k < m$

(Claim 1) $\Leftrightarrow m \geq k+1$

$\Rightarrow j_{k+1} \in \mathcal{O}$

By Lemma 1, $f(i_k) \leq f(j_k)$



→ Consider the situation right after Greedy algo adds i_k to S. (Note: i_k is last interval added to S).

$\Longrightarrow j_{k+1} \in R$ $\{$ j_{k+1} doesn't conflict with $i_k, i_{k-1}, \vdots, i_1$ $\}$

(Ex) $\Longrightarrow R \neq \emptyset \Rightarrow$ Greedy algo cannot terminate

$\Rightarrow$ contradicts ∎

Pf (idea) of Lem 1: By induction on ℓ

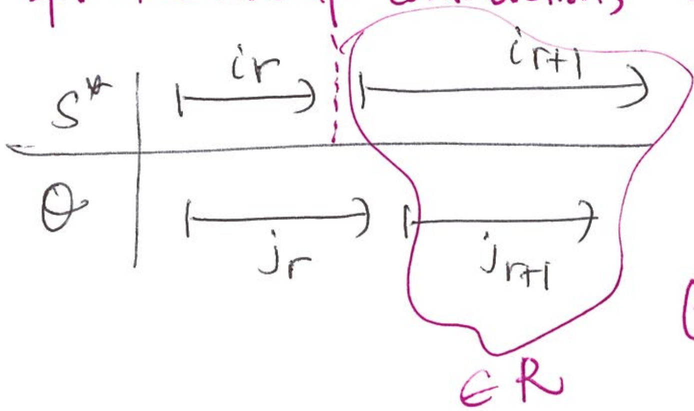
Base case: $\ell = 1$. $f(i_1) \leq f(j_1)$ ← By algo defn $i_1 = 1$ & $f(1)$ is smallest finish time

I.H. Assume for some $r \geq 1$

$\forall\ 1 \leq \ell \leq r$, $f(i_\ell) \leq f(j_\ell)$

I.S. $f(i_{r+1}) \leq f(j_{r+1})$

for the sake of contradiction, assume $f(i_{r+1}) > f(j_{r+1})$



→ Consider the algo right after i_r is added to S

$\Longrightarrow i_{r+1} \in R$

$j_{r+1} \in R$ (Ex)

$\Longrightarrow$ As $f(j_{r+1}) < f(i_{r+1})$ algo cannot pick i_{r+1}

$\Longrightarrow$ contradicts $i_{r+1} \in S^*$ ∎