Sep 15

THEOREM: For any input (M, W, $2n$ pref lists) $n = |M| = |W|$ the GS algo outputs a stable matching

$\Rightarrow$ **COROLLARY**: Every input to the stable matching problem, has a stable matching.

Pf: Follows from the Theorem.

Pf of Theorem

→ Let's say S is the output of GS algo for some arbitrary input. (want to argue: S is a stable matching).

Lemma 1: For every input, GS algo terminates in $\leq n^2$ iterations.

Lemma 2: S is a perfect matching

Lemma 3: S has no instability

Lemmas 1+2+3 $\Rightarrow$ THEOREM.

Pf (idea) of Lemma 1: In each iteration, there is a new proposal $w \to m$ $w \in W$, $m \in M$

$\Rightarrow$ #iterations = #proposals $\leq$ #pairs $(w, m) = |W \times M| = n^2$
$w \in W$
$m \in M$

$\Rightarrow$ #iterations $\leq n^2$ ■

Obs 0: S is matching

Obs 1: Once a man gets engaged, he keeps getting engaged to better women

Obs 2: If w proposes to m after m', $m' > m$ in L_w

Lemma 4: If at the end of an iteration, w is free
$\Rightarrow$ w has NOT proposed to all men

Pf idea of Lemma 2: Proof by contradiction (use Obs 0, Lemmas 1 + 4, Algo definition)

Pf details: Assume that S is not a perfect matching
$\Longrightarrow$ $\exists$ a free woman w
$\xrightarrow[\text{by Lem 4}]{}$ $\boxed{\exists \text{ a man } m \text{ that } w \text{ has not proposed to}}$ ——— $(*)$

(by Obs 0) + Alg def.

By Lemma 1, GS terminates $\xrightarrow[\text{(by alg. def.)}]{}$ all free women have proposed to ALL men.
$\Rightarrow$ contradicts $(*)$ ∎

Pigeon-hole principle: If $\leq n-1$ pigeons into n holes $\Rightarrow$ $\exists$ at least one empty hole.

Pf details of Lemma 4: Assume $\exists$ a free woman w who has proposed to ALL men.
$\xrightarrow[\text{(by Obs 1) + Algo def}]{}$ all n men engaged ——— $(*)$

Since w is free $\Rightarrow$ $\leq n-1$ women are engaged.
$\xrightarrow[\text{PHP}]{}$ ≥ 1 man is not engaged
$\Rightarrow$ $\leq n-1$ men ~~men~~ are engaged
$\Rightarrow$ contradicts $(*)$.

PHP
hole :: man
pigeon :: woman
assign :: engaged