Oct 26

Lemma: $T(n) \leq c \cdot n \cdot \log_2 n + cn$ $\quad (\leq O(n \cdot \log n))$

$\Rightarrow$ MergeSort runs in $O(n \cdot \log n)$ time

Some remarks:

1. $O(n \cdot \log n)$ is the best known upper bound on general sorting algs
2. Can do faster ($O(n)$ time) if domain of a_i's is of size $O(n)$ (there is a T/F q on Piazza)
3. Can do faster runtime for "almost" sorted inputs
4. Any comparison based algo needs to make $\Omega(n \cdot \log n)$ comparisons.

Strategies for solving recurrences

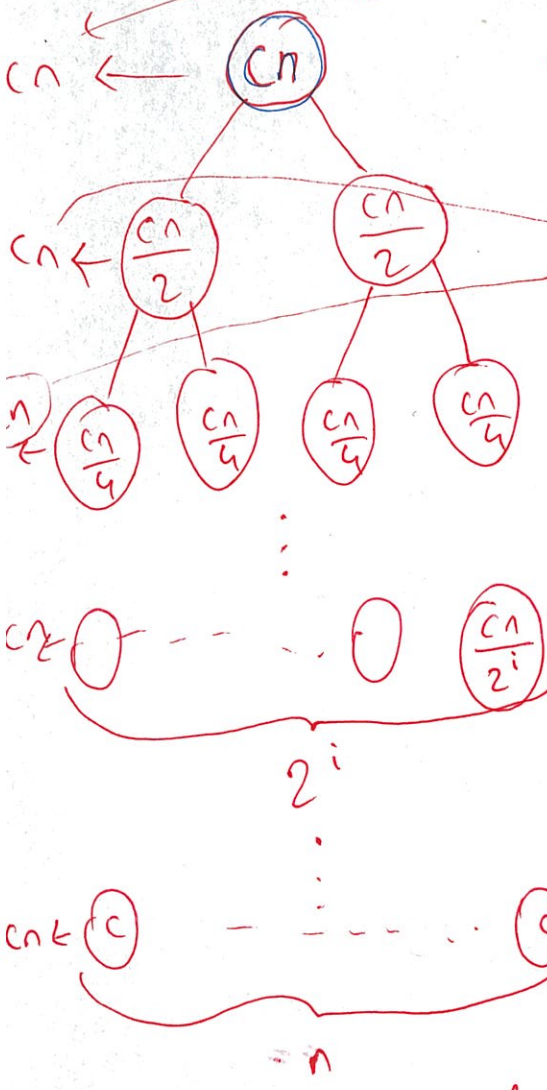
1. "Unroll" the recurrence & identify a pattern
2. Guess the answer & verify by induction on n

$$T(n) \leq \begin{cases} c & \text{if } n=1 \\ cn + 2 \cdot T\left(\frac{n}{2}\right) & \text{o.w.} \end{cases}$$

Goal: $T(n) \leq cn \log_2 n + cn$

Assume that n is a power of 2

Strategy 1: "unroll" and "use pattern"



level 0 $T(n) \leq cn + 2 \cdot T\left(\frac{n}{2}\right)$

$\leq cn + 2\left(\frac{cn}{2} + 2 \cdot T\left(\frac{n}{4}\right)\right)$

$= cn + cn + 4 \cdot T\left(\frac{n}{4}\right)$

$\leq cn + cn + 4 \cdot \left(\frac{cn}{4} + 2 \cdot T\left(\frac{n}{8}\right)\right)$

$= cn + cn + cn + 8 \cdot T\left(\frac{n}{8}\right)$

Contribution from level i: $\left(\frac{n}{2^i} > 1\right)$

$2^i \cdot \frac{cn}{2^i} = cn$

$\frac{n}{2^l} = 1 \iff n = 2^l$

$\iff l = \log_2 n$

$\Rightarrow T(n) \leq cn \cdot (\#\text{ levels})$

$= cn(l+1)$

$= cn(\log_2 n + 1)$

$= cn \cdot \log_2 n + cn$ ∎

Strategy 2: Guess $T(n) \leq c \cdot n \cdot \log_2 n + cn$ — (*)

Verify by induction on n

Base case: $n = 1$ need to show $c \leq c \cdot 1 \cdot \log_2 1 + c$

$= c$

$c \leq c$

(*) holds for $n \leftarrow n/2$

I.H. $T\left(\frac{n}{2}\right) \leq c \cdot \frac{n}{2} \cdot \log_2 \frac{n}{2} + \frac{cn}{2}$

$= \frac{cn}{2}\left(\log_2 \frac{n}{2} + 1\right)$

$= \frac{cn}{2}\left(\log_2 n - \log_2 2 + 1\right)$

$= \frac{cn}{2} \cdot \log_2 n$

I.S. By recurrence

$T(n) \leq cn + 2 \cdot T\left(\frac{n}{2}\right)$

I.H. $\rightarrow \leq cn + 2 \cdot \left(\frac{cn}{2} \cdot \log_2 n\right)$

$= cn + cn \cdot \log_2 n$

$= cn\left(\log_2 n + 1\right)$ ∎