Nov 9

<u>Simplified problem</u> Only want to compute the <u>value</u> of an optimal solution.

<u>Assume:</u> $f_1 \le f_2 \le \dots \le f_n$

<u>Def:</u> $OPT(j)$ = value of an optimal solution for $\underbrace{\text{instance } [j]}_{(s_1, f_1, v_1), \dots, (s_j, f_j, v_j)}$

$j \in [n]$

<u>Goal:</u> Compute $OPT(n)$

<u>Assume:</u> $OPT(0) = 0$

<u>Def:</u> Let O_j be <u>an</u> optimal solution for $[j]$

$\Rightarrow OPT(j) \underset{(*)}{=} v(O_j) = \sum_{i \in O_j} v_i$

<u>Goal:</u> $OPT(j) = \max\{\underbrace{OPT(j-1)}_{j \notin O_j}, \; v_j + \underbrace{OPT(p(j))}_{j \in O_j}\}$

<u>Case 1:</u> $j \notin O_j$

<u>Claim 1:</u> O_j is also optimal for $[j-1]$

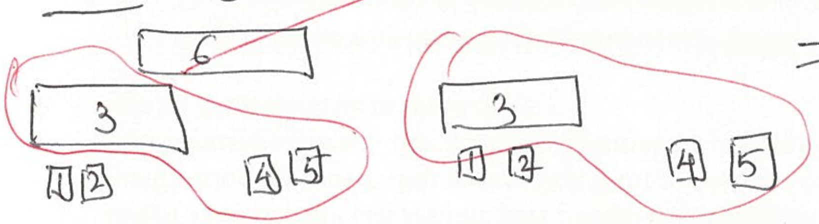
$\Rightarrow OPT(j) \underset{(*)}{=} v(O_j) \underset{\text{by Claim 1}}{=} OPT(j-1)$

<u>Pf (idea) of Claim 1:</u> By contradiction

Assume O_j is NOT optimal for $[j-1]$

$\Rightarrow \exists$ a valid schedule $O' \subseteq [j-1]$ s.t. $v(O') > v(O_j)$

<u>Note:</u> O' is also a valid solution for $[j]$

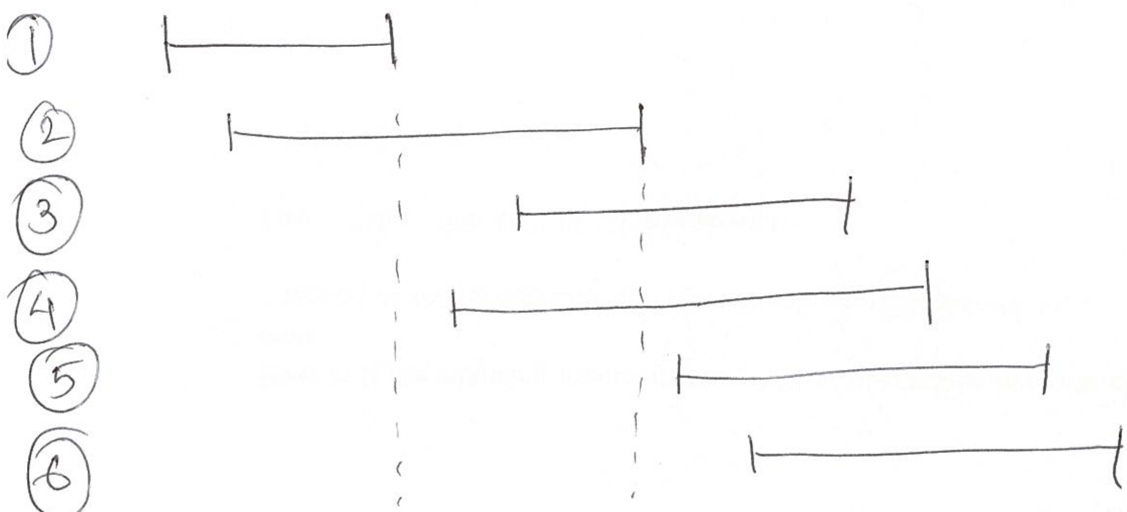


$\Rightarrow O'$ is a valid schedule for $[j]$

AND $v(O') > v(O_j)$

$\Rightarrow O_j$ is <u>not</u> optimal for $[j] \Rightarrow$ contradiction ∎

Case 2: $j \in O_j$ Def: $p(j) = \begin{cases} \text{largest index } i < j \text{ s.t. } i \,\&\, j \text{ do not conflict} \\ = 0 \quad \text{o/w} \end{cases}$



$p(1) = 0$

$p(2) = 0$

$p(3) = 1$

$p(4) = 1$

$p(5) = 2$

$p(6) = 2$

NOTE: (i) $p(j)+1, \ldots, j-1$ conflict with j

[By def] (ii) $1, \ldots, p(j)$ do not conflict w/ j

(recall: $f_1 \le f_2 \le \cdots \le f_n$)

$\Rightarrow$ If $j \in O_j$, then after picking j, the remaining sub-problem is $\underline{1 \ldots p(j)}$

<u>Claim 2:</u> $O_j \setminus \{j\}$ is also optimal for $[p(j)]$ set diff.

$\Rightarrow OPT(j) = v(O_j) = v_j + \underline{v(O_j \setminus \{j\})}$

$= v_j + OPT(p(j))$

by Claim 2

<u>Pf (idea) of Claim 2:</u> By contradiction. Assume $\exists\, O' \subseteq [p(j)]$ that is a valid schedule for $[p(j)]$ AND $\underline{v(O') > v(O_j \setminus \{j\})}$

<u>Note:</u> $O' \cup \{j\}$ is a valid schedule for $[j]$

follows from

$v(O' \cup \{j\}) = v_j + v(O')$

$> v_j + v(O_j \setminus \{j\}) = v_j + v(O_j) - v_j$

$= v(O_j) = OPT(j)$

$\Rightarrow$ contradicts optimality of O_j for $[j]$ ∎

Ex1: Compute $p(1), \dots, p(n)$ in $O(n \log n)$ time

Ex2: Any algo to compute $p(1), \dots, p(n)$ needs $\Omega(n \log n)$ comparison.