THM 1: 3-SAT $\leq_p$ IS $\Longrightarrow$ IS is NP complete.

3-SAT is NP-complete + IS $\in$ NP

Pf: We'll show for any 3-SAT formula Φ $\xrightarrow[\text{poly time}]{}$ G; m $C_1, \ldots, C_m$

s.t Φ is sat. $\Longleftrightarrow$ G has an IS of size $\geq m$.

Dec 7

$\Phi \to$ [pre. proc] $\xrightarrow{G, m}$ [Algo of IS] $\xrightarrow{0/1}$ same 0/1

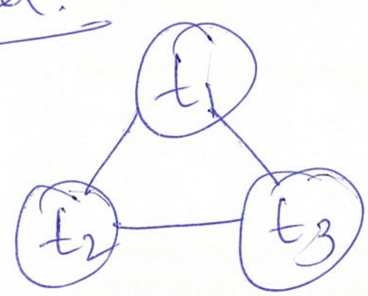
Reduction idea   Use a "gadget"

2 equiv ways of looking at 3-SAT

→ Assign 0/1 to each $X_1, \ldots X_n$ s.t the assignment satisfies ≥ 1 literal in each clause

→ pick one literal from each clause (set to true) s.t you do NOT pick conflicting clauses (not pick X_i and $\bar{X_i}$ for the same i)

Gadget: $C = t_1 \vee t_2 \vee t_3$



IS: $\{t_1\}$ $\{t_2\}$ $\{t_3\}$

even IS $\equiv$ picking a ~~clause~~ literal for C.

Redux: Given $\Phi = C_1, \ldots, C_{\underline{m}} \to (G, \underline{m})$

Φ is satisfiable $\Longleftrightarrow$ G has an IS of size $\geq m$

Step 1: Replace each C_i by its own Δ

Step 2: Add all edges between two nodes labeled with X_i & $\overline{X_i}$ for some i

$n = 4$

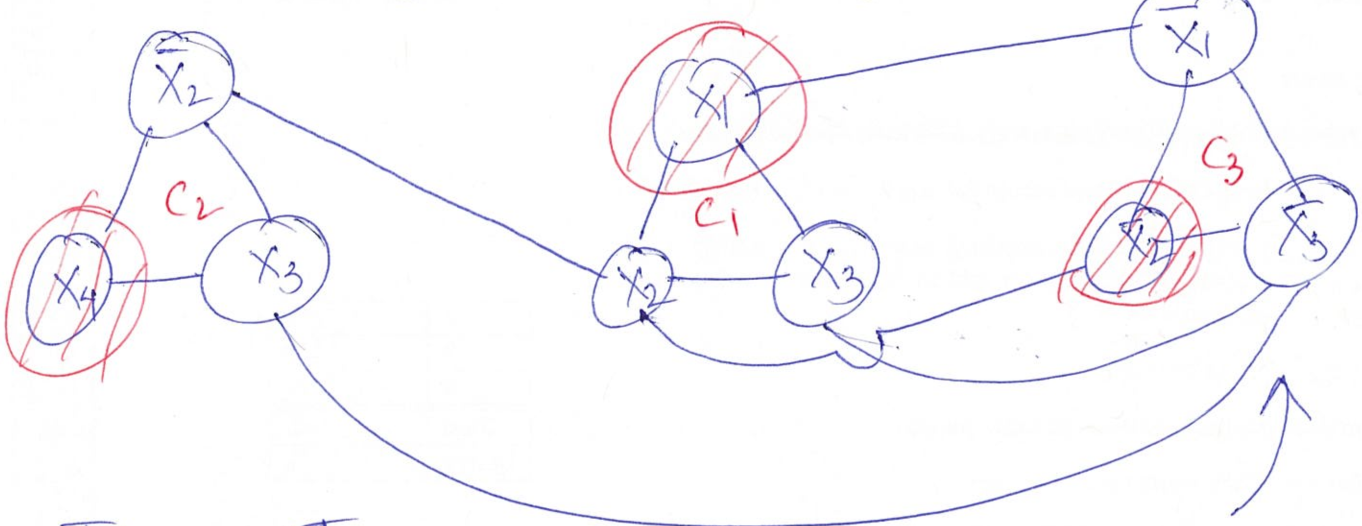
$C_1 = X_1 \vee X_2 \vee X_3$

$C_2 = \overline{X_2} \vee X_3 \vee X_4$

$C_3 = \overline{X_1} \vee \overline{X_2} \vee \overline{X_3}$

$IS = \{X_1, X_4, \overline{X_2}\}$ (C_1, C_2, C_3)

$= \{1, 0, ?, 1\}$ (X_1, X_2, X_4)



THM: Φ is satisfiable $\iff$ G has an IS of size $\geq m$ $(= m)$

(Pf in book.)

Recall: X is NP-complete if

(1) $X \in NP$

(2) $\forall Y \in NP,\ Y \leq_p X$ — (*)

Lemma 2: Let Y be NP-complete + $X \in NP$

If $Y \leq_p X \Rightarrow X$ is NP-complete

(Implicit argm If $Z \leq_p Y$, $Y \leq_p X$

$\Rightarrow Z \leq_p X$)

Lemma 1: Let X be an NP-complete problem.

$$X \in P \iff P = NP$$

Pf: $(\Leftarrow)$ If $P = NP$ then as $X \in NP$ $\Rightarrow X \in P$

as X is NP comp.

$(\Rightarrow)$ Assume $X \in P$

As X is NP-complete $\forall Y \in NP$ $Y \leq_p X$

but as $X \in P \Rightarrow Y \in P \; \forall Y \in NP$

$\left.\begin{array}{l}\Rightarrow NP \subseteq P \\ \text{we know } P \subseteq NP\end{array}\right\} \equiv P = NP$ ∎

THM 2: 3-SAT is NP-complete (Pf: book)

COR 1: IS is NP-complete (THM2 + THM1 + IS $\in NP$, Y = 3-SAT, X = IS)

COR 2: VC is NP-complete (Lemma 2: IS is NP-complete + VC $\in NP$ + IS $\leq_p$ VC)

General strategy to prove X is NP-complete

Step 1: $X \in NP$ (X = IS)

Step 2: Identify an NP-complete problem Y (Y = 3-SAT)

Step 3: $Y \leq_p X$

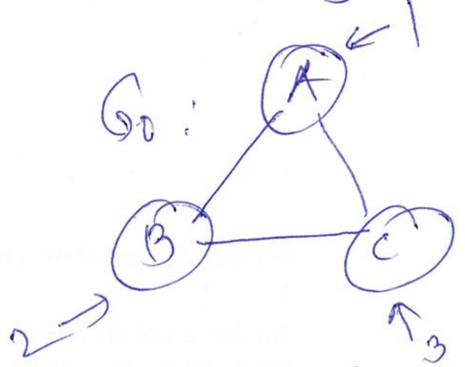
k-colorability (k-coloring)

$G = (V, E)$

Def: k-coloring $C : V \to \{1, .., k\}$

s.t. $\forall (u, w) \in E \quad C(u) \neq C(w)$

Def (k-colorability / k-coloring problem)



Input: $G = (V, E)$; k

O/P:
- 1 if G is k-colorable ($\exists$ a k-coloring of G)
- 0 o/w

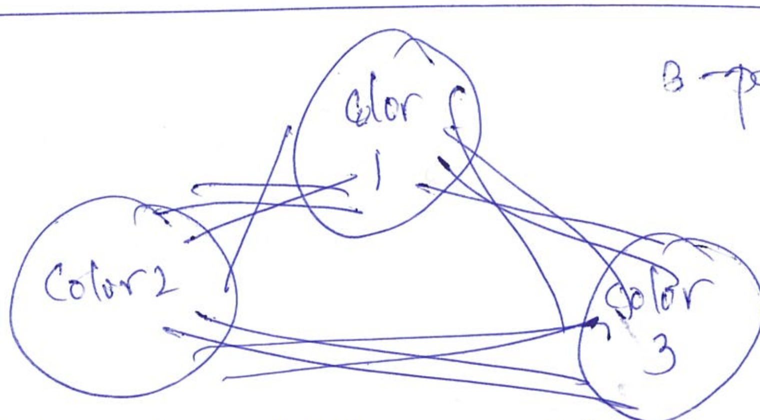
$\exists$ 3-coloring
no 2-coloring

E.g: $G_0 ; 3$ ✓ $\quad G_0 ; 2$ ✗

Claim 1: k-colorability $\in NP$

Pf idea: witness : $C : V \to \{1, .., k\}$

3-colorable graph

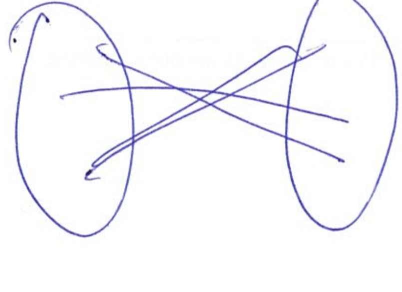


3-partite graph.

k-colorable graph
$\equiv$ k-partite graph

THM: 3-coloring is NP-complete.

2-colorability $\in P$



Bipartiteness