SEP 12

THEOREM: For every input (n, M, W, $2n$ pref lists) the GS outputs a stable matching (!) $|M| = |W| = n$

⇒ **COROLLARY!** Every input to the stable matching problem has a stable matching.

Pf: follows from THEOREM.

Pf of THEOREM

→ Say S is the o/p of the GS algo on an arbitrary input.

Want to argue: S is a stable matching

Lemma 1: For every i/p, GS terminates ($\leq n^2$ iterations)

Lemma 2: S is a perfect matching

Lemma 3: S has NO instability

Lemmas 1+2+3 ⇒ THM.

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Pf (idea) of Lemma 1: (Care package details)

(By algo def.)
In each iteration of the algo, there is a new proposal $w \mapsto m$ $w \in W$, $m \in M$

$\Rightarrow$ # iterations = # proposals $\leq$ # pairs (w, m) $= |W \times M| = |W| \cdot |M| = n \cdot n = n^2$
$w \in W$
$m \in M$

$\Rightarrow$ # iterations $\leq n^2$

Obs 0: S is a matching

Obs 1: Once $m \in M$ gets engaged he remains engaged to better woman

Obs 2: If w proposes to m after m' $\Rightarrow$ $m' > m$ in L_w

Lemma 4: If at the end of an iteration, w is free then w has NOT proposed to all men.

Pf idea of Lemma 2: (Goal: S is a perfect matching)

Pf by contradiction (Use Obs 0, Lemmas 1 + 4, Algo def.)

Pf details Assume S is NOT a perfect matching

$\xRightarrow{\text{Obs 0 + algo def.}}$ $\exists$ a free woman $\underline{w}$

$\xRightarrow{\text{By Lemma 4}}$ $\exists$ a man m that w has <u>not</u> proposed to yet —— (*)

By Lemma 1, GS has terminated.

$\xRightarrow{\text{Algo def}}$ all free woman have proposed to all men

$\Rightarrow$ w has proposed to all men

$\longrightarrow$ contradicts (*) ■

Pigeon hole principle: If $\leq n-1$ pigeons are placed in n holes $\Rightarrow$ $\exists$ one empty hole.

Pf idea of Lem 4: Pf by contradiction (Use Obs 1, PHP, Algo Def)

Pf details of Lemma 4: Assume $\exists$ free woman w who has proposed to all men

$\xRightarrow{\text{Algo def, Obs 1}}$ at termination all n men are engaged —— (#)

Since w is free $\xrightarrow[\text{algo dif}]{}$ $\le n-1$ women who are engaged

$\xrightarrow[\text{PHP}]{}$ ≥ 1 man who is free

PHP
hole :: man
pigeon :: woman

$\Rightarrow$ $\le n-1$ men are engaged

$\Rightarrow$ contradicts (#)