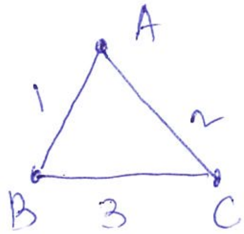
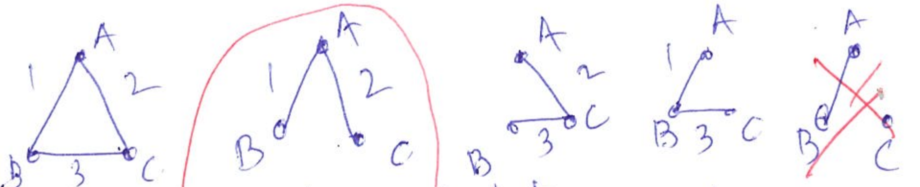
Oct 23





cost $= 1+2+3 = 6$

cost $= 1+2 = 3$ $E' = \{(A,B), (A,C)\}$ min cost

cost $= 2+3 = 5$

cost $= 1+3 = 4$

Input: $G = (V, E)$

connected   undirected

$\forall e \in E, \quad c_e \geq 0$ this is for convenience only (algos work w/ -ve edge costs as well)

cost

Output: $E' \subseteq E$ s.t.

(i) $T = (V, E')$ is connected

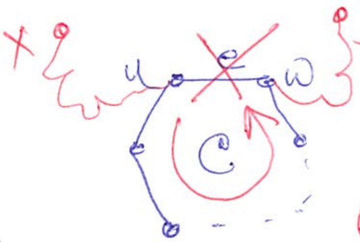
(ii) $c(T) = \sum_{e \in T} c_e$ is minimized

Minimum Spanning Tree (MST)

PROP: Let $c_e > 0 \ \forall e \in E$, then every optimal solution $T = (V, E')$ is a tree.

Pf (details) By contradiction. Let $T = (V, E')$ be an optimal solution (*) s.t. T is not a tree.

$\Longrightarrow$ (as T is connected) $\exists$ a cycle C in T. Let $e = (u, w)$ be an edge in C



$E'' \subseteq E'$

Goal: Show another spanning subgraph $T' = (V, E'')$ s.t. $\text{cost}(T') < \text{cost}(T) \Rightarrow T$ is not optimal   connected   contradiction

$\Rightarrow$ Delete e from T i.e. $T' = (V, E' \setminus \{e\})$ set difference

Claim 1: $c(T') < c(T)$. $\quad c(T') = c(T) - c_e < c(T)$ as $c_e > 0$

Claim 2: T' is connected

Let $x \neq y \in V$

Case 1: $\exists$ an x-y path that does not use e ✓

Case 2: All x-y paths use edge e

If a path uses $e = (u, w)$, take the "scenic path" from u to w in C ✓

$\Rightarrow (x, y)$ still connected in T' ∎

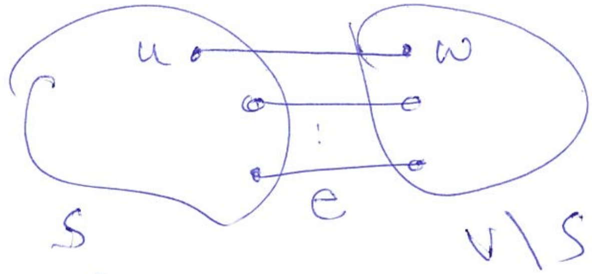
Assume: All c_e's are distinct (we'll later remove this.)

CUT PROPERTY Lemma

For all cuts $(S, V\setminus S)$ s.t. $S \neq \emptyset$ and $V \setminus S \neq \emptyset$

$\equiv$

$S \neq V$



Consider all crossing edges $(u, w) \in E$ s.t. $u \in S$, $w \notin S$

Let e be the crossing edge with minimum cost.

$\Rightarrow e$ is in ALL MSTs for G.

Proof of correctness (idea): Prim/ ~~K~~ Kruskal

every edge added by the algo is cheapest crossing edge for some cut.