Nov 27 RECALL

Lemma: If G has no negative cycle, then $\forall s$, $\exists$ a shortest s-t path that is simple.

$\Rightarrow \exists$ shortest s-t path with $\leq n-1$ edges in it.

Def: $OPT(s, i)$ = cost of shortest s-t path of length $\leq i$ (ie uses $\leq i$ edges)

$s \in V$ [# sub-problems $= n^2$)

$0 \leq i \leq n-1$

Goal: Compute $OPT(s, n-1)$ $\forall s \in V$

Bellman-Ford algo

Recursion

$OPT(t, 0) = 0$, $\forall u \neq t \in S$, $OPT(u, 0) = \infty$

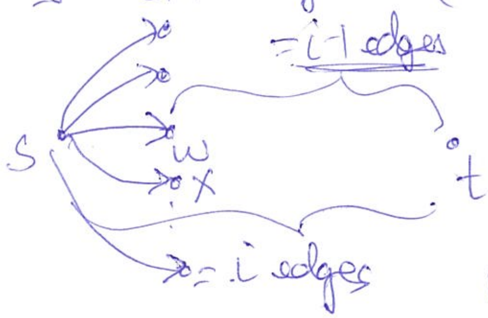
Consider $OPT(u, i)$, $i > 0$ with $\leq i$ edges

Case 1: $\exists$ a shortest s-t path that actually uses $\leq i-1$ edges

$OPT(u, i) = OPT(u, i-1)$

Case 2: All shortest s-t paths of length $\leq i$ edges uses $= i$ edges.

$\Rightarrow \exists$ an edge (s, w) that comes first



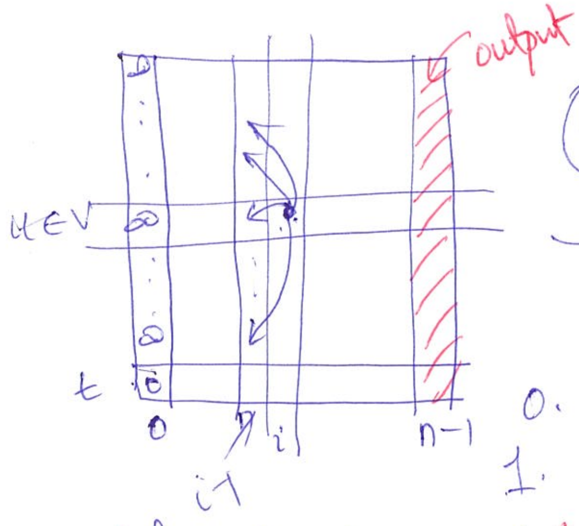
$OPT(s, i) = c_{(s,w)} + OPT(w, i-1)$

In general:

$$OPT(s, i) \min_{x:\,(s,x) \in E} \left\{ c_{(s,x)} + OPT(x, i-1) \right\}$$

Overall: $i > 0$

$$\boxed{OPT(u, i) = \min \left\{ OPT(u, i-1), \min_{x:\,(u,x) \in E} \left\{ c_{(u,x)} + OPT(x, i-1) \right\} \right\}}$$



$M[u,i] = OPT(u,i)$

Q3 Ordering : increasing order of columns

Bellman-Ford

0. Allocate an $n \times n$ matrix M — $O(n^2)$
1. $M[t,0] \leftarrow 0$, $M[u,0] \leftarrow \infty \ \forall u \neq t$ ← $O(n)$
2. $O(n^3)$ { for $i = 1 .. n-1$ } $\leq n^2$
   for $u \in V$
   $M[u,i] \leftarrow \min\{ M[u,i-1], \min_{x:(u,x)\in E} \{ c_{(u,x)} + \cancel{OPT(x)}\ M[x,i-1] \} \}$ ← $O(n)$
3. Return $M[s, n-1] \ \forall s \in V$ ← $O(n)$

Overall: $O(n^3)$