Dec 6

Satisfiability / SAT problem

General: SAT formula on variables $X = \{x_1, \ldots, x_n\}$ (Boolean)

↳ AND clauses

↳ OR of literals

↳ literal $\in \{x_i, \overline{x_i}\}$

E.g. Φ_0: $(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_2 \vee \overline{x_3})$

Generally: $C_1 \wedge C_2 \wedge \cdots \wedge C_m$ C_i: clauses

$\equiv C_1, C_2, \ldots, C_m$

Clause C_i: OR of literals $t_1 \vee t_2 \vee \cdots \vee t_\ell$

each $t_i \in \{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$

Assignment: $a: X \to \{0, 1\}$ (0 = F, 1 = T)

$n = 3$

x_1	$=0$	1	0
x_2	$=0$	1	0
x_3	$=0$	1	1

① $\Phi_0\begin{pmatrix} x_1 & x_2 & x_3 \\ 0, & 0, & 0 \end{pmatrix} = (0 \vee \overline{0}) \wedge (\overline{0} \vee \overline{0}) \wedge (0 \vee \overline{0})$

$= (0 \vee 1) \wedge (1 \vee 1) \wedge (0 \vee 1)$

$= 1 \wedge 1 \wedge 1 = 1$

$\Rightarrow (0,0,0)$ is a satisfy assignment for Φ_0

② $\Phi_0(1,1,1) = (1 \vee \overline{1}) \wedge (\overline{1} \vee \overline{1}) \wedge (1 \vee \overline{1})$

$= (1 \vee 0) \wedge (0 \vee 0) \wedge (1 \vee 0)$

$= 1 \wedge 0 \wedge 1 = 0$

$\Rightarrow$ (1,1,1) is not a satisfying assignment for Φ_0

Def: An assignment satisfies a SAT formula Φ, if Φ evaluates to TRUE/1 given the assignment

$\equiv$ ~~the assi~~ ALL ~~ead~~ clauses evaluate to TRUE on the assignment.

SAT problem / Satisfiability problem

Input: Φ

Output: 1 if $\exists$ a satisfying assignment to Φ

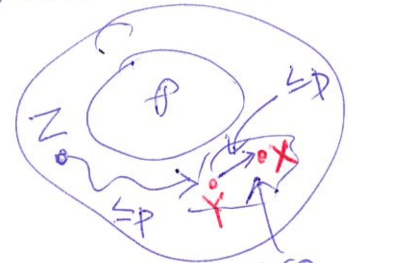
Eg. i/p(1): $(\overset{1}{X_1} \vee \bar{X}_2 \vee X_4), (\bar{X}_1 \vee \bar{X}_3 \vee \overset{1}{X_5}), (\overset{1}{X_2} \vee \bar{X}_3 \vee X_6)$ } 3-SAT formula

o/p: 1 ~~TRUE~~ {as $(1, 1, ?, ?, 1, ?)$ is a satisfy assign)

i/p(2): $(X_1 \vee \ldots)$, $(\bar{X}_1 \vee \ldots)$

o/p: 0 $\equiv X_1, \bar{X}_1 = 0$

3-SAT formula is a SAT formula $C_1, \ldots, C_m$ s.t each C_i has exactly 3 literals



NP-complete

3-SAT problem:

Input: 3-SAT formula Φ

Output: 1 iff Φ is satisfiable.

Lemma 1: 3-SAT $\in$ NP (Ex.)

Pf (idea): Witness: an assignment

Recall: X is NP-complete if (i) $X \in NP$ (ii) every $Y \in NP$ $Y \le_p X$

THM 1 3-SAT is NP-complete. ← see book for proof

saying $\boxed{X \text{ is NP-complete} \equiv X \text{ is "hard"}}$

Lemma 2: If Y is NP-complete & $Y \le_p X$ & $X \in NP$
$\Rightarrow$ X is NP-complete.

(Pf idea: $\underset{NP \ni}{Z} \le_p Y$, $Y \le_p X \Rightarrow Z \le_p X$

General strategy to prove a new problem X is NP-complete

(i) $X \in NP$

(ii) Reduce a <u>known</u> NP-complete problem Y in poly time to X. $(Y \leq_p X)$

In "practice" Y is often 3-SAT.

<u>Goal:</u> IS is NP-complete.

<u>Seen:</u> IS $\in$ NP

<u>THM2:</u> 3-SAT $\leq_p$ IS

$\Big\} \Rightarrow$ IS is NP-complete.

<u>Pf (idea):</u> Given 3-SAT Φ formula $= C_1, \ldots, C_m$

$\longrightarrow G; m$

s.t. Φ is satisfiable $\Longleftrightarrow$ G has an IS of size $\geq m$.

$\Phi \rightarrow \boxed{\text{pre-proc}} \xrightarrow{G,m} \boxed{\text{Algo for IS}} \xrightarrow{0/1}$ same 0/1

<u>Reduction idea:</u> Use a "gadget"

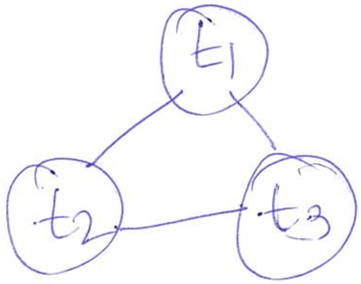
2 equiv ways of looking at 3-SAT: Φ is satisfiable

(1) Assign 0/1 to each of $x_1, \ldots, x_n$ s.t. the assignment satisfies ≥ 1 literal in each clause

(2) Pick one literal from each clause $C_1, \ldots, C_m$ s.t you do <u>NOT</u> pick ~~x~~ x_i & $\overline{x_i}$ for the same i.

Gadget:

$C_j = t_1 \vee t_2 \vee t_3$



IS: $\emptyset$

$\{t_1\}$, $\{t_2\}$, $\{t_3\}$

Each non-empty IS $\equiv$ picking a literal for C

Redux:

Step 1: For each clause generate it's Δ

Step 2: Add an edge between x_i & $\overline{x_i}$ for all i

↳ Intuition: An IS in G will not pick x_i & $\overline{x_i}$

↓ graph G

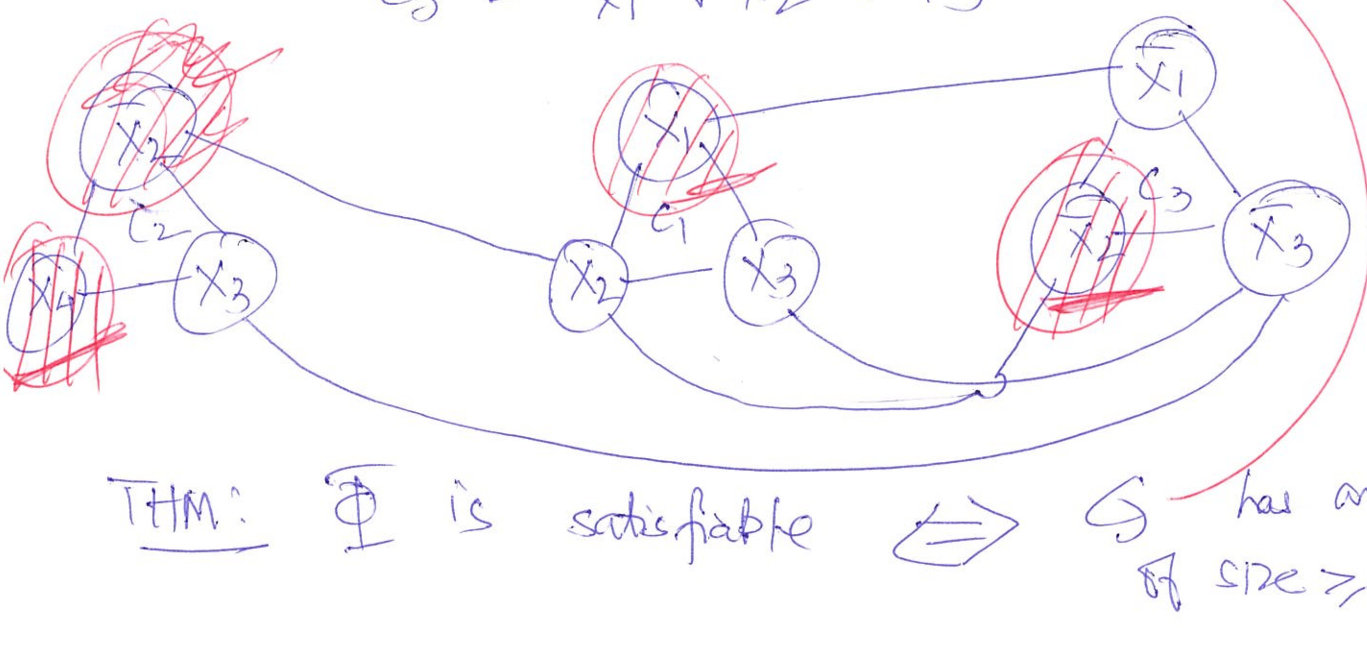
$n = 4$

$m = 3$

$C_1 = x_1 \vee x_2 \vee x_3$

$C_2 = \overline{x_2} \vee x_3 \vee x_4$

$C_3 = \overline{x_1} \vee \overline{x_2} \vee \overline{x_3}$



IS $= \{x_1, \overline{x_2}, x_4\}$

$x_1 = 1$

$x_2 = 0$

$x_4 = 1$

...

THM: Φ is satisfiable $\iff$ G has an IS of size $\geq m$