Oct 20

LEMMA 3: Any optimal schedule Θ' with 0 idle time (but non-zero # inversions), then we can convert Θ' to Θ s.t. (such that) (1) Θ is also optimal (2) Θ has 0 idle time AND (3) Θ has no inversions.

Notation: (1) idle(S) = idle time of S (2) #inv(S) = # inversions in S.

Assume: #inv$(\Theta') > 0$.

Pf idea: By "exchange argument"

$$\Theta' = \Theta_1 \longrightarrow \Theta_2 \longrightarrow \cdots \Theta_i \longrightarrow \Theta_{i+1} \longrightarrow \cdots \Theta_m \longrightarrow \Theta_{m+1} = \Theta$$

idle$(\Theta') = 0$

#inv$(\Theta') > 0$

idle$(\Theta) = 0$

#inv(Θ)

Properties: (i) If idle$(\Theta_i) = 0 \Rightarrow$ idle$(\Theta_{i+1}) = 0$

(ii) #inv$(\Theta_i) > 0 \Rightarrow$ #inv$(\Theta_{i+1}) =$ #inv$(\Theta_i) - 1$

(iii) $L(\Theta_{i+1}) \le L(\Theta_i)$

Idea: Apply this transformation (say m time) s.t. #inv$(\Theta_{m+1}) = 0$

Note: (i) idle$(\Theta_1) = 0 =$ idle$(\Theta_2) =$ idle$(\Theta_3) = \cdots =$ idle(Θ_m) = idle(Θ)

(ii) #inv$(\Theta_{m+1}) =$ #inv$(\Theta) = 0$

(iii) $L(\Theta) = L(\Theta_{m+1}) \le L(\Theta_m) \le L(\Theta_{m-1}) \le \cdots \le L(\Theta_2)$
$\le L(\Theta_1) = L(\Theta')$

$\Rightarrow$ as Θ' is optimal $L(\Theta) = L(\Theta')$ ∎

NOTE: #inv$(\Theta_1) \le$ ~~max~~ total # pairs $= \binom{n}{2} \le n^2$

$\Rightarrow \boxed{m \le n^2}$

We want: $\forall i$ $\quad Q_i \longrightarrow Q_{i+1}$ s.t. if idle$(Q_i)=0$ & #inv$(Q_i)>0$

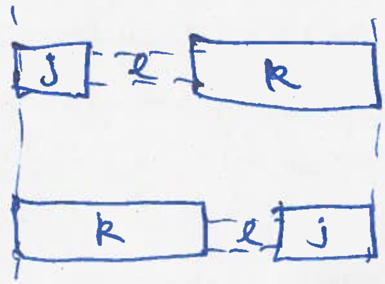
$\Rightarrow$ (i) idle$(Q_{i+1})=0$ (ii) #inv(Q_{i+1}) = #inv$(Q_i)-1$
(iii) $L(Q_{i+1}) \leq L(Q_i)$

#inv$(Q_i)>0$ $\Rightarrow$ $\exists$ an inversion (j, k)

Q_i:

Swap j&k

Q_{i+1}:



d_k d_j

<u>Obs:</u> #inv(Q_{i+1}) < #inv(Q_i)

Ex: Pf by case analysis.

<u>BUT!</u> max lateness could increase.

<u>Special:</u>

d_k d_j is good for max lateness ✓

<u>Claim:</u> Special is not so special!

Will argue:

<u>Property (a):</u> [as #inv$(Q_i)>0$] $\exists$ an inversion (j, k)

s.t.

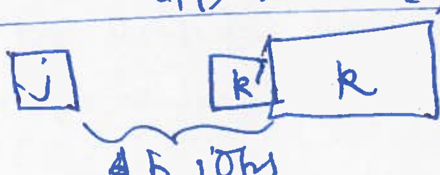
d_k d_j

<u>Property (b):</u> Swap j&k to get Q_{i+1} $\Rightarrow$ idle$(Q_{i+1})=0$

$\Rightarrow$ #inv(Q_{i+1}) = #inv$(Q_i)-1$

<u>Property (c):</u> $L(Q_{i+1}) \leq L(Q_i)$

<u>Property (a):</u>



Assume $\exists$ jobs k' between j & k

$d_{k'}$ d_k d_j

<u>Easy case:</u> (k', k) is an inversion ✓

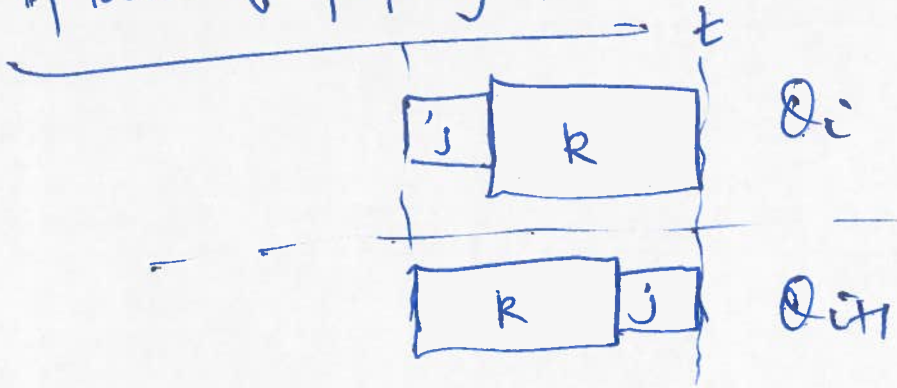
<u>other Case</u>: (k', k) is NOT an inversion

$\Rightarrow d_{k'} \leq d_k$ BUT $d_j > d_k$ as (j, k) is an inversion

$\Rightarrow d_j > d_{k'}$ $\Rightarrow$ (j, k') is an inversion.

⇒ Apply above $\le b$ times ✓

Pf idea of property (C)



$\forall\, i \ne j$ or k

l_i does not change.

$l'_i = l_i$

lateness in Q_{i+1}

lateness in Q_i

Need: $\max(l'_j, l'_k) \le \max(l_j, l_k)$

Easy case: (A): $l'_k \le l_k \le \max(l_k, l_j)$

~~BUT~~ Other case: ~~If~~ is possible $l'_j > l_j$

BUT: $l'_j = t - d_j - 1$ [assume $l'_j > 0$.]

$< t - d_k - 1$ [note: $d_k < d_j$

$\iff -d_j < -d_k$

$= l_k$

$\Rightarrow l'_j < l_k \le \max(l_k, l_j)$ — (B)

(A) + (B) $\Rightarrow \max(l'_j, l'_k) \le \max(l_j, l_k)$