Oct 24

Lemma 3: $\exists$ an optimal schedule with 0 idle time and 0 #inversions

without loss of generality

Lemma 3': Any optimal soution θ' with 0 idle time (but possibly non-zero #inversions) $\Rightarrow$ construct another optimal solution θ with (1) 0 idle time and (2) 0 # inversions.

Notation: (1) Idle(S) = idle time of schedule S
(2) #inv(S) = # inversions in ______

Assume: $\#inv(\theta') > 0$.

Pf idea: By "exchange argument"

$\theta' = \theta_1 \mapsto \theta_2 \quad \cdots \theta_i \mapsto \theta_{i+1} \cdots \cdots \theta_m \mapsto \theta_{m+1} = \theta$

$idle(\theta') = 0$
$\#inv(\theta') > 0$

$idle(\theta) = 0$
$\#inv(\theta) = 0$

Properties (i) $idle(\theta_i) = 0 \Rightarrow idle(\theta_{i+1}) = 0$
(ii) $\#inv(\theta_i) > 0 \Rightarrow \#inv(\theta_{i+1}) = \#inv(\theta_i) - 1$
(iii) $L(\theta_{i+1}) \le L(\theta_i)$

$\to$ Idea: Apply this (say m times) s.t $\#inv(\theta_{m+1}) = 0$

Note: (i) $\Rightarrow idle(\theta') = 0 = idle(\theta_1) = idle(\theta_2) \cdots\cdots$
$= idle(\theta_m) = idle(\theta_{m+1}) = idle(\theta)$
$\Rightarrow 0 = idle(\theta') = idle(\theta)$

(ii) $\#inv(\theta_{m+1}) = \#inv(\theta) = 0 \quad \left[\#inv(\theta') \le \#\text{pairs } (i,j) = \binom{n}{2} \le n^2 \right.$

(iii) $\Rightarrow L(\theta) = L(\theta_{m+1}) \le L(\theta_m) \le L(\theta_{m-1}) \cdots \le L(\theta_{i+1}) \le L(\theta_i)$
$\le \cdots L(\theta_2) \le L(\theta_1) = L(\theta')$

$\Rightarrow L(\theta) \le L(\theta') \underset{\text{as } \theta' \text{ is optimal}}{\Longrightarrow} L(\theta) = L(\theta')$ ∎

<u>We want:</u> $\Theta_i \longmapsto \Theta_{i+1}$ s.t. $idle(\Theta_i)=0$, $\#inv(\Theta_i)>0$

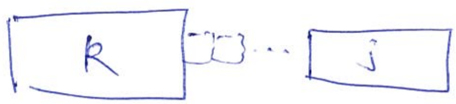
(i) $idle(\Theta_{i+1})=0$ (ii) $\#inv(\Theta_{i+1}) = \#inv(\Theta_i)-1$ (iii) $L(\Theta_{i+1}) \le L(\Theta_i)$

$\#inv(\Theta_i)>0 \Rightarrow \exists$ an ~~in~~version (j,k)

Θ_i:

Q: How do we get rid of (j,k)? Swap j & k

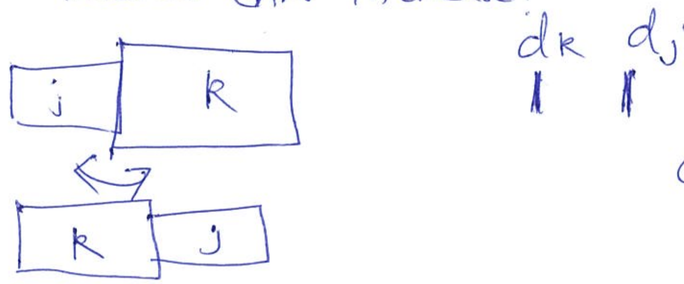
Θ_{i+1}:



<u>Ex:</u> $\#inv(\Theta_{i+1}) < \#inv(\Theta_i)$

<u>BUT:</u> max lateness CAN increase!

<u>Special case!</u>



#inversions ↓

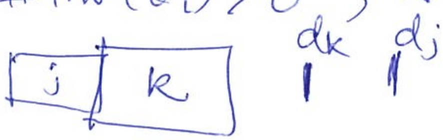
Can show: max lateness cannot increase

<u>Claim:</u> Special case is not special!

Argue 3 things:

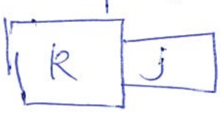
<u>Property (a)</u> $\#inv(\Theta_i)>0$, $\exists$ an inversion (j,k) st

Θ_i:



<u>Property (b)!</u> Swap j & k to get $\Theta_{i+1} \Rightarrow \#inv(\Theta_{i+1}) = \#inv(\Theta_i)-1$

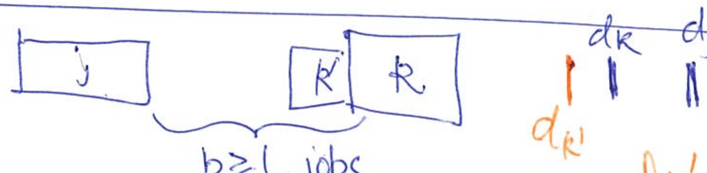
Θ_{i+1}



Also $idle(\Theta_{i+1})=0$

<u>Property (c)!</u> $L(\Theta_{i+1}) \le L(\Theta_i)$

<u>Property (a)!</u> (j,k) inversion



<u>Case 1!</u> (k',k) is an inversion

<u>Case 2!</u> (k',k) is NOT an inversion $\Rightarrow d_{k'} \le d_k$ → $d_j \ge d_{k'}$, j comes but before k'

Consider (j,k') As $d_j \ge d_k \ge d_{k'} \Rightarrow (j,k')$ is an inversion

$\Rightarrow$ Apply the above argument $\le b$ times ■

Pf (idea) of property (c)

d_k d_j